Contents

Introduction

Objective and Application

The objective of the Capital Asset Pricing Model ("CAPM") is to

1. generate a diversified market portfolio;
2. apply the portfolio to price individual assets and portfolios;
3. define idiosyncratic and systematic risk of individual assets;
4. measure hedging and diversifying characteristics of various assets;

Assumptions

The following assumptions are used to derive the CAPM formulas.

1. No transaction costs;
2. Infinitely divisible stocks;
3. No personal income tax for investors;
4. No monopoly power of each individual investor in the stock market;
5. A utility function of the form $U(\boldsymbol{r}, \Sigma)$ (function that depends on the expected return and variance matrix of the returns). The inputs of the model are represented by the following objects:
    a. Expected rate of return $\boldsymbol{r}$**;**
    b. Returns variance matrix Σ
    c. Representative investor utility function (typically of the form $U(\boldsymbol{x}) = \boldsymbol{rx} - \frac{\gamma}{2}\boldsymbol{x}\Sigma\boldsymbol{x}$**;**
    d. Risk-free rate r_{rf}
6. Unlimited short sales (no constraints on the investor positions). An alternative constrained CAPM model assumes that investors cannot enter into short positions;;
7. Unlimited borrowing / lending at risk-free rate;
8. Homogeneity of investor expectations with respect to the expected returns and returns variance matrix;
9. All assets are marketable.

CAPM Model (Two-Period)

The two-period CAPM model is derived by estimating the demand for each asset, assuming fixed supply of each asset and constructing prices such as demand equals to supply for each asset. It is assumed that the price of each asset share in period $t = 0$ is normalized to $P_i = 1$ (the shares of the traded firms are selected so that the price of each share is normalized to one).

With the normalized prices, each investor maximizes the return on the portfolio conditional on the controlled portfolio volatility. The investor problem and supply/demand equilibrium are estimated below.

Solving Asset Demand Problem

Consider an investor with $1 wealth who considers allocation of the wealth between a set of assets. The assets are indexed with i and denoted as $\mathcal{A}_i$. Suppose that one period vector of return and variance matrix of the assets are denoted as $\boldsymbol{r}$ and Σ. Allocation of wealth into different assets is denoted as $\boldsymbol{x}$. Investor optimization problem is formulated as follows:

$$\begin{cases} max_x \left[\boldsymbol{rx} - \frac{\gamma}{2}\boldsymbol{x}\Sigma\boldsymbol{x}\right] \\ \boldsymbol{1x} = 1 \end{cases}$$

All vectors in the math notations are displayed with bold symbols. Vector $\boldsymbol{1}$ is vector with element at each position equal to 1. Suppose that α is Lagrange multiplier of the optimization problem. Then the Lagrangian is equal to

$$\mathcal{L} = \boldsymbol{rx} - \frac{\gamma}{2}\boldsymbol{x}\Sigma\boldsymbol{x} + \alpha(1 - \boldsymbol{1x})$$

The system of first order conditions and constraint are presented below.

$$\begin{cases} \mathcal{L}_x = 0\text{:}\ \boldsymbol{r} + \alpha\boldsymbol{1} = \gamma\Sigma\boldsymbol{x} \\ \boldsymbol{1x} = 1 \end{cases}$$

After solving the first-order conditions, the following solution is obtained: $\boldsymbol{x} = \frac{1}{\gamma}\Sigma^{-1}(\boldsymbol{r} + \alpha\boldsymbol{1})$ and $\frac{1}{\gamma}[\boldsymbol{1}\Sigma^{-1}\boldsymbol{r} + \alpha\boldsymbol{1}\Sigma^{-1}\boldsymbol{1}] = 1$. The following notations are used in the notes:

$$\begin{cases} A = \boldsymbol{1}\Sigma^{-1}\boldsymbol{1} \\ B = \boldsymbol{1}\Sigma^{-1}\boldsymbol{r} \\ C = \boldsymbol{r}\Sigma^{-1}\boldsymbol{r} \end{cases}$$

Lagrangian multiplier is equal to

$$\alpha = \frac{\gamma - B}{A}$$

and optimal portfolio is

$$\boldsymbol{x} = \frac{\gamma - B}{A\gamma}\Sigma^{-1}\boldsymbol{1} + \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{r}$$

Suppose that

$$\boldsymbol{x}_{min} = \frac{1}{A}\Sigma^{-1}\mathbf{1}$$

and

$$\boldsymbol{x}_{\mathcal{M}} = \frac{1}{B}\Sigma^{-1}\boldsymbol{r}$$

Then the optimal investment portfolio is represented as

$$\boldsymbol{x} = \left(1 - \frac{B}{\gamma}\right)\boldsymbol{x}_{min} + \frac{B}{\gamma}\boldsymbol{x}_{\mathcal{M}}$$

This is a two fund separation theorem that states that each efficient portfolio can be represented as a linear of two portfolios which do not depend on the investor preferences γ: $\boldsymbol{x}_{min}$ and $\boldsymbol{x}_{\mathcal{M}}$.

As $\gamma \to \infty$, $\boldsymbol{x} \to \boldsymbol{x}_{min}$. That is, $\boldsymbol{x}_{min}$ is a minimum variance portfolio, a portfolio which is a solution of the investor's problem that constructs a portfolio with minimum variance.

Alternatively, the efficient portfolio is presented as follows:

$$\boldsymbol{x} = \boldsymbol{x}_{min} + \frac{B}{\gamma} \times (\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min})$$

Note that

$$Cov(\boldsymbol{x}_{min}, \boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}) = \boldsymbol{x}_{min}\Sigma(\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}) = \frac{1}{A} \times \mathbf{1} \times (\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}) = 0$$

since the constraint $\mathbf{1}\boldsymbol{x}_{\mathcal{M}} = 1$ and $\mathbf{1}\boldsymbol{x}_{min} = 1$ holds for both portfolios. The minimum variance portfolio is orthogonal to $\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}$ portfolio.

Characteristics of the two portfolios are calculated below.

1. Expectation and variance of $\boldsymbol{x}_{min}$ portfolio.

$$E[\boldsymbol{x}_{min}] = \boldsymbol{r}\boldsymbol{x}_{min} = \frac{B}{A}$$

$$Var[\boldsymbol{x}_{min}] = \boldsymbol{x}_{min}\Sigma\boldsymbol{x}_{min} = \frac{1}{A^2}(\mathbf{1}\Sigma^{-1}\mathbf{1}) = \frac{1}{A}$$

2. Expectation and variance of $\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}$ portfolio.

$$E[\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}] = \boldsymbol{r}(\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}) = \frac{C}{B} - \frac{B}{A} = \frac{AC - B^2}{AB} = \frac{D}{AB}$$

   where $D = AC - B^2 = (\mathbf{1}\Sigma^{-1}\mathbf{1}) \times (\boldsymbol{r}\Sigma^{-1}\boldsymbol{r}) - (\mathbf{1}\Sigma^{-1}\boldsymbol{r})^2 = 1 - Cor^2(\Sigma^{-1}\mathbf{1}, \Sigma^{-1}\boldsymbol{r}) \geq 0$.

$$Var[\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}] = (\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min})\Sigma^{-1}(\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}) = \left(\frac{\boldsymbol{r}}{B} - \frac{\mathbf{1}}{A}\right)\Sigma^{-1}\left(\frac{\boldsymbol{r}}{B} - \frac{\mathbf{1}}{A}\right) = \frac{C}{B^2} - \frac{2}{A} + \frac{1}{A}$$
$$= \frac{D}{AB^2}$$

3. Expectation and variance of $\boldsymbol{x}$ portfolio.

$$E[\boldsymbol{x}] = \boldsymbol{r}\boldsymbol{x} = \frac{B}{A} + \beta \times \frac{D}{AB} = \frac{1}{A}\left(B + \frac{D}{\gamma}\right)$$

$$Var[\boldsymbol{x}] = \boldsymbol{x}\Sigma^{-1}\boldsymbol{x} = Var[\boldsymbol{x}_{min}] + \beta^2 Var[\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}] = \frac{1}{A} + \beta^2 \times \frac{D}{AB^2} = \frac{1}{A}\left(1 + \frac{D}{\gamma^2}\right)$$

The following relationship holds between $\mu = E[\boldsymbol{x}]$ and $\sigma^2 = Var[\boldsymbol{x}]$. Substituting out $\frac{1}{\gamma}$ as $\frac{1}{\gamma} = \frac{A\mu - B}{D}$ we obtain:

$$\sigma^2 = \frac{1}{A}\left(1 + D \times \left(\frac{A\mu - B}{D}\right)^2\right) = \frac{1}{A}\left(1 + \frac{1}{D} \times (A\mu - B)^2\right)$$

The equation is called the efficiency frontier for the set of underlying assets. The efficiency frontier shows different (μ, σ^2) combinations that are selected by investors with different risk preference γ parameters.

CAPM Equilibrium

Demand for each asset for a given expected return and return covariance matrix was derived as

$$\boldsymbol{x} = \left(1 - \frac{B}{\gamma}\right)\boldsymbol{x}_{min} + \frac{B}{\gamma}\boldsymbol{x}_{\mathcal{M}} = \frac{\gamma - B}{A\gamma}\Sigma^{-1}\mathbf{1} + \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{r}$$

In equilibrium,

$$\boldsymbol{x} = \boldsymbol{w}$$

or

$$\frac{\gamma - B}{A\gamma}\Sigma^{-1}\mathbf{1} + \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{r} = \boldsymbol{w}$$

where

$$\begin{cases} A = \mathbf{1}\Sigma^{-1}\mathbf{1} \\ B = \mathbf{1}\Sigma^{-1}\boldsymbol{r} \end{cases}$$

Solving for $\boldsymbol{r}$ we obtain

$$\boldsymbol{r} = \frac{B - \gamma}{A}\mathbf{1} + \gamma\Sigma\boldsymbol{w}$$

The supply $\boldsymbol{w}$ represents also the market index. The above function shows that the rate of return must be a linear function of the security covariance with the market portfolio, $\Sigma\boldsymbol{w}$. Suppose that $\boldsymbol{\pi}$ is an arbitrary portfolio. Then the return on portfolio is described by the following equation:

$$\boldsymbol{\pi r} = \frac{B - \gamma}{A}\mathbf{1} + \gamma\boldsymbol{\pi}\Sigma\boldsymbol{w}$$

Suppose that $\boldsymbol{\pi_{min}}$ is an arbitrary portfolio that is not correlated with the market portfolio $\boldsymbol{w}$.

$$\boldsymbol{\pi_{min}}\Sigma\boldsymbol{w} = \mathbf{0}$$

Then the return on the portfolio is equal to

$$\boldsymbol{\pi_{min} r} = \frac{B - \gamma}{A}\mathbf{1} = r_{min}$$

Alternatively, suppose that $\boldsymbol{\pi_M} = \boldsymbol{w}$ is the market portfolio. Then

$$\boldsymbol{\pi_M r} = \frac{B-\gamma}{A}\mathbf{1} + \gamma \boldsymbol{w}\Sigma\boldsymbol{w} = r_{min} + \gamma\sigma_M^2 = r_M$$

or, equivalently,

$$r_M - r_{min} = \gamma\sigma_M^2$$

The equilibrium equation can be represented now in the traditional CPAM form

$$\boldsymbol{r} = \frac{B-\gamma}{A}\mathbf{1} + \gamma\Sigma\boldsymbol{w} = r_{min} + \frac{\Sigma\boldsymbol{w}}{\sigma_M^2} \times [r_M - r_{min}] = r_{min} + \boldsymbol{\beta} \times [r_M - r_{min}] \quad \text{(CAPM)}$$

In general, return r_{min} can be selected arbitrary in the model. Based on the selected parameter r_{min}, and risk parameter γ, market return r_M and market returns of the traded assets $\boldsymbol{r}$ are estimated based on the CAPM model. Conditional on the estimated prices, the two funds $\boldsymbol{x}_{min}$ and $\boldsymbol{x}_{\mathcal{M}}$ are estimated including the related efficiency frontier and the funds risk/return parameters.

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CAPM Summary

The formulas below summarize the CAPM model.

CAPM Model Inputs

The model is estimated based on the following inputs:

i. **Market portfolio**. The market portfolio of acquired companies $i = 1, \dots n$. Each company is described by its value $V_i = E_i + D_i$ which we estimate as the sum of the related firm debt and equity values. The price per share of each company in period $t = 0$ is normalized to one and V_i is assumed to be the number of issued shares. We assume that the shares are issued on the total value of the company (not its equity value). The price and return on debt and equity are estimated as a price and related return on the derivative of the company total value.

ii. **Historical company values**. For each company in the selected portfolio, we obtain the historical values of the company market debt and equity price, D_{it} and E_{it}. The historical sample of company values is estimated then as $V_{it} = E_{it} + D_{it}$.

iii. **Market portfolio weights**. The weight of each security in the market portfolio: $\boldsymbol{w} = (w_1, \dots w_n)$. Each weight is estimated as $w_i = \frac{V_i}{\sum_j V_j}$. Each asset value is estimated as of the CAPM valuation date.

iv. **Historical returns.** The historical returns on the company values are estimate as follows: $r_{i,t} = \frac{V_{i,t+1}-V_{i,t}}{V_{i,t}} + \frac{c_{i,t}+d_{i,t}}{V_{i,t}} = \left(\frac{D_{i,t+1}-D_{i,t}}{D_{i,t}} + \frac{c_{i,t}}{D_{i,t}}\right) \times \frac{D_{i,t}}{V_{i,t}} + \left(\frac{E_{i,t+1}-E_{i,t}}{E_{i,t}} + \frac{d_{i,t}}{E_{i,t}}\right) \times \frac{E_{i,t}}{V_{i,t}}$, where is coupon payment and $d_{i,t}$ is dividend payment in period $[t, t+1]$. Under constant debt yield rate structure, the return on debt $y = \frac{D_{i,t+1}-D_{i,t}}{D_{i,t}} + \frac{c_{i,t}}{D_{i,t}}$ is equal to the yield rate y.

v. **Returns covariance matrix**. The return covariance matrix Σ is constructed based on historical return rates $r_{i,t}$.

vi. **Risk-free return**. Risk-free return is denoted as r_f and is estimated based either on Treasury rates or swapped Libor rates.

vii. **Risk-averse parameter**. We set utility risk-aversity parameter γ to some arbitrary positive value.

Investor Problem

$$\boldsymbol{x} = \frac{\gamma - B}{A\gamma}\Sigma^{-1}\mathbf{1} + \frac{1}{\gamma}\Sigma^{-1}\boldsymbol{r} = \left(\mathbf{1} - \frac{B}{\gamma}\right) \times \frac{\Sigma^{-1}\mathbf{1}}{A} + \frac{B}{\gamma} \times \frac{\Sigma^{-1}\boldsymbol{r}}{B} = \frac{\Sigma^{-1}\mathbf{1}}{A} + \frac{B}{\gamma} \times \left[\frac{\Sigma^{-1}\boldsymbol{r}}{B} - \frac{\Sigma^{-1}\mathbf{1}}{A}\right]$$

Where coefficients are calculated as

$$\begin{cases} A = \mathbf{1}\Sigma^{-1}\mathbf{1} \\ B = \mathbf{1}\Sigma^{-1}\boldsymbol{r} \\ C = \boldsymbol{r}\Sigma^{-1}\boldsymbol{r} \end{cases}$$

and

$$D = AC - B^2 = 1 - Cor^2(\Sigma^{-1}\mathbf{1}, \Sigma^{-1}\boldsymbol{r}) \geq 0$$

Two-fund separation theorem

$$\boldsymbol{x} = \boldsymbol{x}_{min} + \frac{B}{\gamma} \times (\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min})$$

where

$$\boldsymbol{x}_{min} = \frac{1}{A}\Sigma^{-1}\mathbf{1}$$

$$\boldsymbol{x}_{\mathcal{M}} = \frac{1}{B}\Sigma^{-1}\boldsymbol{r}$$

For given market returns $\boldsymbol{r}$, each investor optimal portfolio is a combination of two portfolios: $\boldsymbol{x}_{min}$ and $\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}$, which do not depend on the preferences of the investor. Investor preferences effect only the weights applied to each portfolio. Two portfolios, $\boldsymbol{x}_{min}$ and $\boldsymbol{x}_{\mathcal{M}} - \boldsymbol{x}_{min}$, are uncorrelated.

Fund parameters

Fund $\boldsymbol{x}_{min}$

$$E[\boldsymbol{x}_{min}] = \tfrac{B}{A} \text{ and } Var[\boldsymbol{x}_{min}] = \tfrac{1}{A}$$

Fund $\boldsymbol{x}_{\mathcal{M}}$

$$E[\boldsymbol{x}_{\mathcal{M}}] = \tfrac{C}{B} \text{ and } Var[\boldsymbol{x}_{\mathcal{M}}] = \tfrac{C}{B^2}$$

Portfolio $\boldsymbol{x}$

$$E[\boldsymbol{x}] = \tfrac{1}{A}\left(B + \tfrac{D}{\gamma}\right) \text{ and } Var[\boldsymbol{x}] = \tfrac{1}{A}\left(1 + \tfrac{D}{\gamma^2}\right)$$

Efficiency Frontier

If $\mu = E[\boldsymbol{x}]$ and $\sigma^2 = Var[\boldsymbol{x}]$ then

$$\sigma^2 = \frac{1}{A}\left(1 + \frac{1}{D} \times (A\mu - B)^2\right)$$

The function is the parabola with the minimum point going through

$$\mu_{min} = \frac{B}{A}, \qquad \sigma_{min}^2 = \frac{1}{A}$$

Efficiency frontier is estimated for investors with different risk preference γ parameter.

CAPM Equilibrium

Equilibrium prices:

$$\boldsymbol{r} = r_{min} + \boldsymbol{\beta} \times [r_M - r_{min}]$$

where

$$\boldsymbol{\beta} = \frac{\Sigma \boldsymbol{w}}{\sigma_M^2}$$

is security **systematic risk**.

Zero covariance portfolio. An arbitrary portfolio $\boldsymbol{\pi_{min}}$ such that

$$\boldsymbol{\pi_{min}} \Sigma \boldsymbol{w} = \boldsymbol{0}$$

Minimum return. Return on zero covariance portfolio is denoted as

$$r_{min} = \frac{B - \gamma}{A} \boldsymbol{1}$$

Generally, r_{min} is selected arbitrary in the model (or estimated from the observed market prices).

Market return. Market rate of return is estimated as follows

$$r_M - r_{min} = \gamma \sigma_M^2$$

The equation can be used either to estimate the market rate of return r_M for a given γ or to calibrate γ from a given observed market return r_M.

Risk-free asset. If a risk-free asset exists, then formally Σ^{-1} does not exist. If however we assume that the risk-free asset is not correlated with other securities and the volatility of the risk-free asset is $\varepsilon \sim 0$, then $B \sim \frac{r_f}{\varepsilon}$ and $A \sim \frac{1}{\varepsilon}$. The minimum return is approximated then as follows:

$$r_{min} \sim r_f$$

Risk-free rate r_f is set exogenously and is estimated based on Treasury rates, Libor rates, etc.

CAPM Model Output