Theories of Interest Rate

Expectations Theory

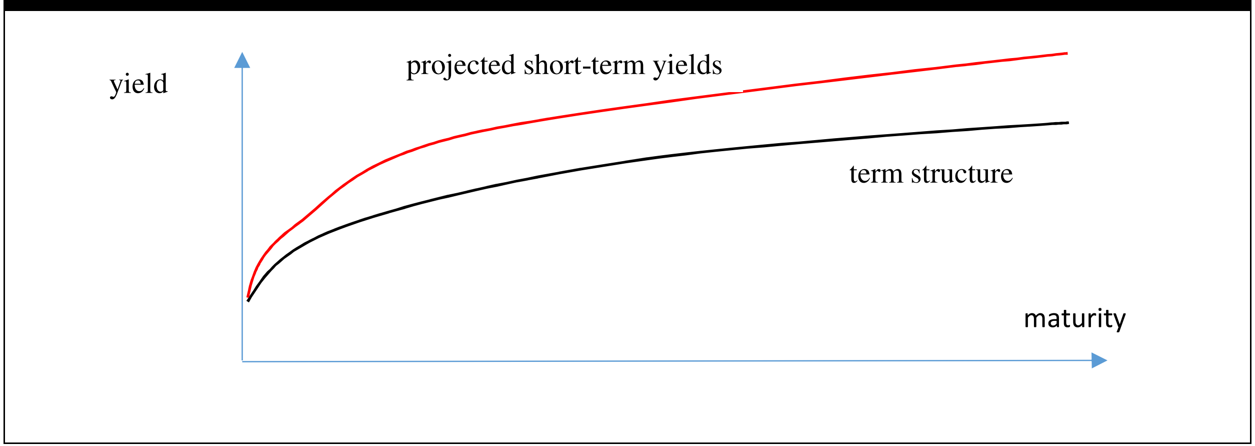
Under the Expectations Theory, bonds with different maturity terms are assumed to be perfect substitutes. Therefore, if a bond with a long-term maturity is replaced by a sequence of bonds with short-term maturities, a return on a long-term bond must be equal to a compounded return on a sequence of short-term bonds. If we denote the yield rate on short-term (three-month) bonds as $y_{t,1}$, the expected yield rate on short-term bonds in a future period s>t as $y^e_{s,1}$, and the yield rate on a bond with maturity τ as $y_{t,\tau}$, then the relationship between the actual and expected short-term yield rates and the yield rates with longer maturities can be described by the following equation:

$$(1 + y_{t,\tau})^{\tau} = (1 + y_{t,1}) \times (1 + y^e_{t+1,1}) \times \ldots \times (1 + y^e_{t+\tau-1,1}) \quad \textbf{(E1)}$$

Based on equation (E1), we can derive the long-term yield rate $y_{t,\tau}$ with maturity term τ using a sequence of current and expected short-term yields. Therefore, under the Expectations Theory, the problem of a term structure forecast is reduced to the problem of properly forecasting future expected short-term yield rates. The complete term structure curve is then constructed using the sequences of short-term yield rates as per equation (E1).

If the expected short-term yields are constant, then the term structure of the yield rates is also constant: $y_{t,\tau} = y_{t,1}$. To match the increasing term structure of the actual yields that are typically observed in the market, the expected short-term yield rates $y^e_{t+1,1}$ must be increasing even faster than the term structure. An illustrative example of the yield term structure and projected short-term yields is shown in the exhibit below.

Increasing term structure and projected expected short-term yield rates



To summarize, rational expectations models produce unrealistic projected expected short-term rates but the generated bond prices are arbitrage free.

Liquidity Premium Theory

Description

Under an alternative Liquidity Premium Theory, bonds with different maturities are not viewed as perfect substitutes and lenders generally require a premium for bonds with longer maturities to compensate for the inflation and interest rate risks. The term structure under the Liquidity Premium Theory is modelled as follows:

$$y_{t,\tau} = y^e_{t,\tau} + \pi_{t,\tau} \tag{E2}$$

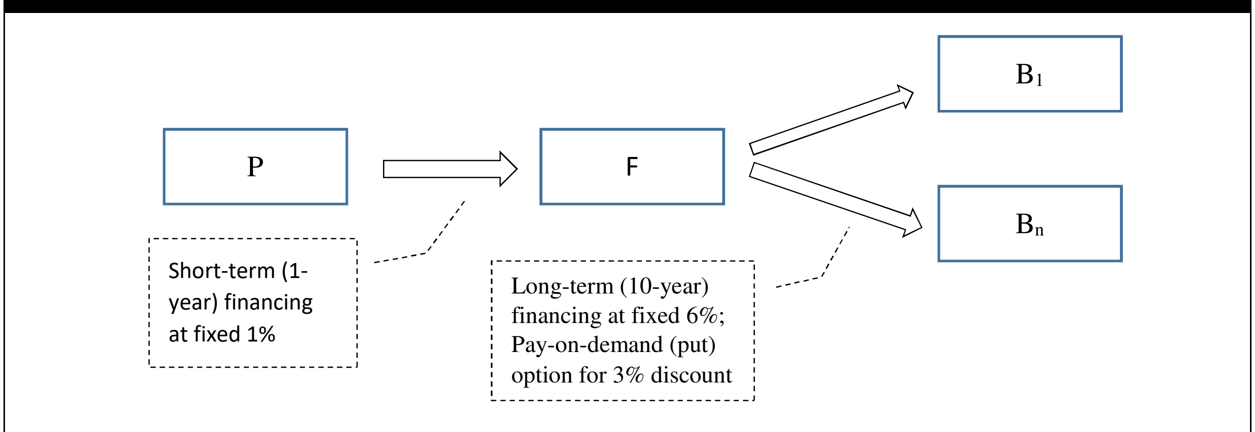
where the term $y^e_{t,\tau}$ is estimated using equation (E1) and the term $\pi_{t,\tau}$ represents a maturity-term premium component of the bond yield rates. Under this approach, the term structure forecast problem is broken down into two separate components: (i) forecasting the expected future short-term rates $y^e_{s,1}$ and calculating the $y^e_{t,\tau}$ components of the medium and long-term rates; and (ii) forecasting the term premiums $\pi_{t,\tau}$ for the medium and long-term rates.

The liquidity premium theory generally produces more realistic projected expected short-term yield rates. However in addition to the short-rate process, the liquidity premium theory requires to model the term premium stochastic process. Moreover certain restrictions must be imposed on the markets to ensure that short-term bonds cannot be used as substitutes to long-term bonds and no-arbitrage opportunities are generated by the term premium.

Term Premium Arbitrage

An example of arbitrage opportunities that may be created based on term premium presence in the bond market is illustrated in the diagram below. Suppose that F is a financing subsidiary that receives funds from the corporate group parent P and lends them to borrowing subsidiaries B_i. Then the subsidiary F can apply the following lending strategy to generate arbitrage profits.

Illustrative example of the term premium arbitrage



To match the funds received from the borrowers B_i and repaid to the parent P, the financing subsidiary F exercises the required number of put options whenever the debt to the parent P is due.

In the example, the financing cost for the subsidiary F is 1% and the interest income is 3% (= 6% - 3%). The profit 2% (= 5% - 3%) is generated due to a risk-free arbitrage produced by the term-premium trading. The example illustrates the following points:

- In the example, 3% put option discount is the discount paid over full 10-year term. Annual discount is approximately 30bps = 3% / 10-year term. As the put option is exercised, the 3% is already is sunk cost that was paid by the lender. Therefore the full 3% (and not the 30bps) is subtracted from the lender's term premium.
- The discount on the put option eliminates partially the term premium on the long-term bond. The arbitrage argument can be used to derive a high-level proxy (lower bound) for the put option:
$$\pi^{put} = \frac{y_T - y_1}{T}$$
where P is the put option price and T is maturity term. For example, if term premium equals 3% ($y_T - y_1 = 3\%$) and maturity term equals 5 years ($t = 5$), then the lower bound for the put option annual discount is 60bps ($0.6\% = \frac{3\%}{5}$). More generally, a 5-year put option annual discount must be at least 20% of the related 5-year term premium.
- The parameters in the put option valuation must be set correctly to ensure that the term structure is matched in the put option model. The put option model must produce the put option value numbers that are consistent with the term structure premium;
- In practice, both call and put options typically have a penalty structure. Call options have make-whole provisions and a step-wise penalty structure after the make whole provision expires. Put options typically have a grace period when they cannot be exercised. Put options can also typically be exercised only at specific dates (for example at annual frequency).

Affine Term Structure Models

Affine term structure models ("ATSM"), or no-arbitrage models, are a popular approach to model interest rates because the term structure derived under this approach is arbitrage-free. The affine structure of interest rates is modelled as follows. At the first step, short-term yield rates$y_{t,1}$ are described using n latent variables $f_{1,t}, \ldots f_{n,t}$ as follows

$$y_{t,1} = \sum_{i=1,..,n} f_{i,t}$$

where each latent factor $f_{i,t}$ is described as follows:

$$\Delta f_{i,t} = k_i \times (\vartheta_i - f_{i,t}) + \sigma_i \sqrt{f_{i,t}} \eta_{i,t} \qquad \textbf{(E4)}$$

Yield rates with longer maturity terms are derived from short-term yield rates assuming that no arbitrage opportunities exist for bonds with longer maturity terms. An arbitrage-free price of a zero-coupon bond with maturity term τ, denoted as $P_{t,\tau}$, is described by the following equation:

$$P_{t,\tau} = e^{\sum_{i=1,\ldots,n}(A_{i,\tau} - B_{i,\tau} f_{i,t})}$$

Respectively, yield rates on longer-term zero-coupon bonds are calculated as follows

$$y_{t,\tau} = -\frac{\ln P_{t,\tau}}{\tau} = -\frac{\sum_{i=1,\dots,n} A_{i,\tau}}{\tau} + \sum_{i=1,\dots,n} \frac{B_{i,\tau}}{\tau} \times f_{i,t} \qquad \text{(E5)}$$

Equation (E4) is a VAR model for the unobserved factors, and equation (E5) describes a linear relationship between the unobserved factors and the yield term structure curve. [1]

The described model described in this section is a generic multi-dimensional affine term structure model. The discussion below focuses specifically on one-dimensional affine term structure model.

Interest Rate Process

The interest rate process models the stochastic behavior of the short-term rate with given observed current rate and given estimated annual volatility. The interest rate process is used to model and evaluate the interest rate based derivative instruments such as loan pre-payment option, loan ay-on-demand option, etc.

Interest Rate Model Examples

General form of interest rate process can be represented as follows:

$$dr_t = \mu(t, r_t) \times dt + \sigma(t, r_t) \times dW_t$$

where different models are summarized in the exhibit below

Model name	Drift term	Diffusion term
Equilibrium Models		
Random Walk (with drift)	$\mu_t = \vartheta$	σ
Vasicek (1977)	$\mu_t = \alpha(\beta - r_t)$	σ
Dothan (1978)	$\mu_t = \alpha r_t$	σr_t
Rendleman – Bartter	$\mu_t = \alpha r_t$	σr_t
Courtadon	$\mu_t = \alpha(\beta - r_t)$	σr_t
Constant Elasticity of Variance (CEV)	$\mu_t = \alpha r_t$	$\sigma_t r_t^{\frac{\gamma}{2}}$
Marsh-Rosenfeld (1983)	$\mu_t = \beta r_t^{-(1-\gamma)} + \alpha r_t$	$\sigma_t r_t^{\frac{\gamma}{2}}$
Cox – Ingersoll – Ross (CIR, 1985)	$\mu_t = \alpha(\beta - r_t)$	$\sigma_t r_t^{\frac{1}{2}}$
Exponential Vasicek (EV)	$\mu_t = r_t \times (\vartheta_t - \alpha_t \ln r_t)$	σr_t

[1] Equations (E4) and (E5) of the ATSM model can be represented as a special case of equation (E3). Parameters A and Λ in the ATSM model specification are described as follows.

1. Matrix A is a diagonal matrix with $A_{i,i} = 1 - k_i$ diagonal elements;
2. Elements $\Lambda_{i,\tau}$ of matrix Λ are calculated as $\Lambda_{i,\tau} = \frac{B_{i,\tau}}{\tau}$, where $B_{i,\tau} = \frac{2\times(e^{-\gamma_i\tau}-1)}{(\gamma_i+k_i+\lambda_i)\times(e^{-\gamma_i\tau}-1)+2\gamma_i}$), $\gamma_i = \sqrt{(k_i+\lambda_i)^2+\sigma_i^2}$, and parameter λ_i is interpreted as the market price of risk.

Model name	Drift term	Diffusion term
No-arbitrage Models		
Ho-Lee (random walk, 1986)	$\mu_t = \vartheta_t$	σ
Hull-White (extended Vasicek, mean reversion, 1990)	$\mu_t = \vartheta_t - \alpha_t r_t$	σ_t
Hull-White (extended CIR, 1990)	$\mu_t = \vartheta_t - \alpha_t r_t$	$\sigma_t r_t^{\frac{1}{2}}$
Black-Derman-Toy (1990)		
Black-Karazinski (1991)	$\mu_t = \vartheta_t - \alpha_t \ln r_t$	σ

Most of the above models allow to derive explicit solution for the bond bullet prices.

Model parameters are estimated by (i) calculating sample mean or (ii) running a simple regression on the drift parameter (in the case of, respectively, (i) random walk or (ii) mean-reversion process) and on constructed residuals (in case of, respectively, (i) homoscedastic and (ii) heteroscedastic model).

Parameter Annualization

Parameters of the interest rate model are generally estimated based on daily data. The interest rate trees are modelled based on quarterly, semi-annual or annual data. The formulas below show how the daily model parameters are converted into equivalent parameters with lower data frequency.

Mean-Reversion Process

Suppose that the process, estimated using daily data, is described by the following equation:

$$\Delta y_s = (\vartheta - \alpha y_t)ds + \sigma dW_s$$

or, equivalently,

$$y_{s+1} = \vartheta ds + (1 - \alpha ds)y_s + \sigma dW_s$$

The relationship between y_{s+1} and y_{s-k} yields can be represented then as follows.

$$\begin{aligned} y_{s+1} &= \big(1 + (1 - \alpha ds)\big)\vartheta ds + (1 - \alpha ds)^2 y_{s-1} + \sigma(dW_s + (1 - \alpha ds)dW_{s-1}) = \cdots \\ &= \big(1 + (1 - \alpha ds) + \cdots + (1 - \alpha ds)^{k-1}\big)\vartheta ds + (1 - \alpha ds)^k y_{s+1-k} \\ &\quad + \sigma\big(dW_s + (1 - \alpha ds)dW_{s-1} + \cdots + (1 - \alpha ds)^{k-1} dW_{s+1-k}\big) \end{aligned}$$

or, equivalently,

$$y_{s+1} = \frac{1 - (1 - \alpha ds)^k}{\alpha ds}\vartheta ds + (1 - \alpha ds)^k y_{s+1-k} + \sigma d\widetilde{W}$$

Value of k is selected as

$$k = \frac{dt}{ds}$$

so that the value of $(1 - \alpha ds)^k$ is approximated as

$$(1 - \alpha ds)^k = \left(1 - \frac{\alpha dt}{dt/ds}\right)^{\frac{dt}{ds}} \sim e^{-\alpha dt}$$

The quarterly mean-reversion process is represented then as follows.

$$y_{s+1} - y_{s+1-k} = \left[\frac{1 - e^{-\alpha dt}}{\alpha dt} \vartheta\right] \times dt - \left[\frac{1 - e^{-\alpha dt}}{dt}\right] dt \times y_{s+1-k} + \sigma d\widetilde{W}$$

where volatility of $\sigma d\widetilde{W}$ is equal to

$$\tilde{\sigma}^2 = \sigma^2 \times ds \times \left(1 + (1 - \alpha ds)^2 + \cdots + (1 - \alpha ds)^{2(k-1)}\right) = \sigma^2 \times ds \times \frac{1 - (1 - \alpha ds)^{2k-1}}{1 - (1 - \alpha ds)^2}$$

and, after approximating $(1 - \alpha ds)^k \sim e^{-\alpha dt}$

$$\tilde{\sigma}^2 = \left[\sigma^2 \times \frac{1 - e^{-2\alpha dt}}{2\alpha dt}\right] \times dt$$

To summarize,

$$\begin{cases} \tilde{\vartheta} & = & \dfrac{1 - e^{-\alpha dt}}{\alpha dt} \vartheta = \dfrac{\tilde{\alpha}}{\alpha} \times \vartheta \\ \tilde{\alpha} & = & \dfrac{1 - e^{-\alpha dt}}{dt} \\ \tilde{\sigma}^2 & = & \sigma^2 \times \dfrac{1 - e^{-2\alpha dt}}{2\alpha dt} \end{cases}$$

or

$$\begin{cases} \tilde{\vartheta} & = & \dfrac{1 - e^{-\hat{\alpha}\frac{dt}{ds}}}{\hat{\alpha} dt} \hat{\vartheta} = \hat{\vartheta} \times \dfrac{\tilde{\alpha}}{\hat{\alpha}} \\ \tilde{\alpha} & = & \dfrac{1 - e^{-\hat{\alpha}\frac{dt}{ds}}}{dt} \\ \tilde{\sigma}^2 & = & \hat{\sigma}^2 \times dt \times \dfrac{1 - e^{-2\hat{\alpha}\frac{dt}{ds}}}{2\hat{\alpha}\dfrac{dt}{ds}} = \hat{\sigma}^2 \times dt \times \dfrac{\tilde{\alpha}}{\hat{\alpha}} \times \dfrac{1 + e^{-\hat{\alpha}\frac{dt}{ds}}}{2} \times ds \end{cases}$$

where $(\hat{\vartheta}, \hat{\alpha})$ are regression parameters estimated using daily yield sample.

Random Walk

Random walk is a special case of mean-reversion process with $\alpha = 0$. The parameters are equal

$$\begin{cases} \tilde{\vartheta} & = & \vartheta \\ \tilde{\alpha} & = & \alpha \\ \tilde{\sigma}^2 & = & \sigma^2 \end{cases}$$

Terminal Distribution Parameters

Parameters of the short-rate terminal distribution are summarized in the exhibit below.

Model name	Drift term	Diffusion term
	Equilibrium Models	
Random Walk	$\mu_T = r_0 + \vartheta T$	$\sigma_T = \sigma\sqrt{T}$
Vasicek (1977)	$\mu_T = \ e^{-\alpha T} r_0 + \frac{1}{\alpha} \times (1 - e^{-\alpha T})\vartheta$ or $\mu_T = \ e^{-\alpha T} r_0 + (1 - e^{-\alpha T}) y^*$ where $y^* = \beta = \frac{\vartheta}{\alpha}$ is the steady state	$\sigma_T = \sigma \times \sqrt{\frac{1 - e^{-2\alpha T}}{2\alpha}}$
Dothan (1978)		
Rendleman – Bartter		
Courtadon		
Constant Elasticity of Variance (CEV)	$\mu_T = \ e^{-\alpha T} r_0 + \frac{1}{\alpha} \times (1 - e^{-\alpha T})\vartheta$	
Marsh-Rosenfeld (1983)		
Cox – Ingersoll – Ross (1985)		
Exponential Vasicek (EV)		
	No-arbitrage Models	
Ho-Lee (random walk, 1986)	$\mu_T = r_0 + \int_0^T \vartheta_s ds = r_0 + \vartheta T$	$\sigma_T = \sigma\sqrt{T}$
Hull-White (extended Vasicek, mean-reversion, homoscedastic, 1990)	$\mu_T = e^{-\alpha T} r_0 + \int_0^T e^{-\alpha(T-s)} \vartheta_s ds$	$\sigma_T = \sigma \times \sqrt{\frac{1 - e^{-2\alpha T}}{2\alpha}}$
Hull-White (extended CIR, mean reversion, heteroscedastic, 1990)		
Black-Derman-Toy (1990)		
Black-Karazinski (1991)		

Zero-Coupon Bond Bullet Prices

The zerto-coupon bullet prices for different models are summarized in the exhibit below.

Model name	Zero-coupon bond bullet price
	Equilibrium Models
Random Walk	$P_T = e^{-r_0 T - \frac{1}{2}\vartheta T^2 + \frac{1}{6}\sigma^2 T^3}$ Equivalently $P_T = A \times e^{-Br_0}$, where $B = T$ and $A = e^{-\frac{1}{2}\vartheta T^2 + \frac{1}{6}\sigma^2 T^3}$
Vasicek (1977)	
Dothan (1978)	
Rendleman - Bartter	
Courtadon	
Constant Elasticity of Variance (CEV)	
Marsh-Rosenfeld (1983)	
Cox – Ingersoll – Ross (1985)	
Exponential Vasicek (EV)	
	No-arbitrage Models
Ho-Lee (1986)	$P_T = e^{-r_0 T - \int_0^T \vartheta_s \times (T-s) ds + \frac{1}{6}\sigma^2 T^3}$ Equivalently $P_T = A \times e^{-Br_0}$, where $B = T$ and $A = e^{-\int_0^T \vartheta_s \times (T-s) ds + \frac{1}{6}\sigma^2 T^3}$
Hull-White (extended Vasicek, 1990)	$P_T = A \times e^{-Br_0}$ where $B = \frac{1-e^{-\alpha T}}{\alpha}$ and $A = e^{\frac{(B-T)\times(\alpha^2\beta - \sigma^2/2)}{\alpha^2} - \frac{\sigma^2 B^2}{4\alpha}}$
Hull-White (extended CIR, 1990)	$P_T = A \times e^{-Br_0}$ where $B = \frac{2(e^{\gamma T}-1)}{(\gamma+\alpha)\times(e^{\gamma T}-1)+2\gamma}$ and $A = \left(\frac{2\gamma e^{(\alpha+\gamma)T/2}}{(\alpha+\gamma)(e^{\gamma T}-1)+2\gamma}\right)^{\frac{2\alpha\beta}{\sigma^2}}$ where $\gamma = \sqrt{\alpha^2 + 2\sigma^2}$
Black-Derman-Toy (1990)	
Black-Karazinski (1991)	

Parameter Calibration

Overview

The yield structure R_T of zero-coupon bond is defined as follows:

$$P_T = e^{-TR_T}$$

or

$$R_T = -\frac{\ln P_T}{T}$$

The yield structure can be estimated from the observed bond prices. The parameters of the model are calibrated so that the theoretical term structure matches as close as possible the estimated term structure. For different models, the equation can be represented as follows:

Random Walk

After substituting $P_T = e^{-r_0 T - \frac{1}{2}\vartheta T^2 + \frac{1}{6}\sigma^2 T^3}$, we get $-\ln P_T = r_0 T + \frac{1}{2}\vartheta T^2 - \frac{1}{6}\sigma^2 T^3$ or

$$TR_T - Br_0 = \frac{1}{2}\vartheta T^2 - \frac{1}{6}\sigma^2 T^3$$

Function $B = T$ does not depend on model parameters and function

$$-\ln A = \frac{1}{2}\vartheta T^2 - \frac{1}{6}\sigma^2 T^3 = A_1 \times \vartheta + A_2 \times \sigma^2$$

is a linear function of parameters ϑ and σ^2, and therefore the parameters can be estimated directly using simple linear regression model. In the equation, $A_1 = \frac{1}{2}T^2$ and $A_2 = -\frac{1}{6}T^3$.

Parameters ϑ and σ^2 can be estimated by running a simple linear regression model that selects the parameters to generate the best fit of the above term structure. The linear regression equation for the yield term structure R_T is described as follows:

Calibration of drift and volatility parameters

$$R_T = r_0 + \frac{1}{2}\vartheta T - \frac{1}{6}\sigma^2 T^2 \qquad \text{(hw.1)}$$

The term structure is matched using the regression model

$$R_T = r_0 + \frac{1}{2}\vartheta T - \frac{1}{6}\sigma^2 T^2 + \varepsilon_t$$

and applying OLS estimation approach. The parameters can be calibrated either using the latest term structure or a panel data of term structures to produce more robust results.

Alternatively, the volatility can be estimated based on the historical short-rate data

Estimation of drift and volatility parameters based on sample statistics

Sample volatility is estimated based on the following short rate representation.

$$dr_t = \vartheta \times dt + \sigma \times dW_t$$

so that $\hat{\sigma}$ is estimated as a sample standard deviation of the $\left\{\frac{dr_t}{\sqrt{dt}}\right\}$ sample:

$$\hat{\sigma} = stdev\left[\frac{dr_t}{\sqrt{dt}}\right]$$

In practical applications, dt is assumed constant and is calculated based on the number of business days during the year: $dt = \frac{1}{250}$. The equation for the sample volatility parameter becomes

$$\hat{\sigma} = \sqrt{250} \times stdev\,[dr_t] \qquad \text{(hw.2)}$$

Conditional on the estimated volatility parameter, the drift parameter ϑ can be estimated as follows.

$$\vartheta = \frac{2}{T} \times (R_T - r_0) + \frac{1}{3}\sigma^2 T \qquad \text{(hw.3)}$$

where T is the maturity term. The drift parameter will generate the term premium $\pi = R_T - r_0$ over the maturity term T.

Initial value r_0:

Original value of r_0 is estimated based on the observed term structure. However after calibrating the volatility and drift parameters and generating the corresponding theoretical term structure (hw.1), the bullet price of the bond may not match the par value. The initial value r_0 is solved for implicitly so that the bullet value of the bod equals to par.

$$r_0 : P^{bullet} = Par\ (100) \qquad \text{(hw.4)}$$

Hull-White (extended Vasicek)

Unconstrained

After substituting $P_T = \left[e^{\frac{(B-T)\times(\alpha^2\beta-\sigma^2/2)}{\alpha^2}-\frac{\sigma^2B^2}{4\alpha}}\right] \times e^{-Br_0}$, we get $-\ln P_T = \frac{\sigma^2B^2}{4\alpha} - \frac{(B-T)\times\left(\alpha^2\beta-\frac{\sigma^2}{2}\right)}{\alpha^2} + Br_0$ or

$$TR_T - Br_0 = \frac{\sigma^2B^2}{4\alpha} - \frac{(B-T)\times\left(\alpha^2\beta - \frac{\sigma^2}{2}\right)}{\alpha^2}$$

The equation can be rewritten as

$$TR_T - Br_0 = -\beta \times (B - T) + \sigma^2 \times \left[\frac{B^2}{4\alpha} + \frac{B-T}{2\alpha^2}\right]$$

The parameters are estimated as follows. For a given value of α, the function $B = \frac{1-e^{-\alpha T}}{\alpha}$ is estimated and the above linear model is estimated. The coefficients σ^2 and β are estimated from the linear model and parameter α is selected to minimize the overall sum of squares in the linear regression. The estimation procedure is reduced to the optimization problem for a non-linear function of α variable.

Constrained

In the constrained version of the model, $\beta = r_0$ so that the steady state is equal to the current yield rate. The equation then becomes:

$$TR_T - Tr_0 = \sigma^2 \times \left[\frac{B^2}{4\alpha} + \frac{B-T}{2\alpha^2}\right]$$

or, formally,

$$B(t) = t,\ \ A_1 = 0,\ \ and\ \ A_2 = \frac{\tilde{B}^2}{4\alpha} + \frac{\tilde{B} - T}{2\alpha^2}$$

Where $\tilde{B} = \frac{1-e^{-\alpha T}}{\alpha}$. The price in the case of constrained model is calculated as follows

$$P_T = e^{-\sigma^2\left[\frac{\tilde{B}^2}{4\alpha}+\frac{\tilde{B}-T}{2\alpha^2}\right]} \times e^{-Tr_0}$$

Hull-White (extended CIR)

After substituting $P_T = \left[\left(\frac{2\gamma e^{(\alpha+\gamma)T/2}}{(\alpha+\gamma)(e^{\gamma T}-1)+2\gamma}\right)^{\frac{2\alpha\beta}{\sigma^2}}\right] \times e^{-Br_0}$, we get $-\ln P_T = -\frac{2\alpha\beta}{\sigma^2} \times \ln\left[\frac{2\gamma e^{(\alpha+\gamma)T/2}}{(\alpha+\gamma)(e^{\gamma T}-1)+2\gamma}\right] + Br_0$ or

$$TR_T - Br_0 = -\frac{2\alpha\beta}{\sigma^2} \times \ln\left[\frac{2\gamma e^{(\alpha+\gamma)T/2}}{(\alpha+\gamma)(e^{\gamma T}-1)+2\gamma}\right]$$

where $\gamma = \sqrt{\alpha^2 + 2\sigma^2}$. The parameters are estimated as follows. For a given pair of values of α and σ, the function $B = \frac{2(e^{\gamma T}-1)}{(\gamma+\alpha)\times(e^{\gamma T}-1)+2\gamma}$ is estimated and then β is estimated from the above linear model. The estimation procedure is reduced to the optimization problem for a non-linear function of (α, σ) variables.

Ho-Lee Model

Calibration for the Ho-Lee model is performed as follows. Unlike the equilibrium models, the no-arbitrage models are estimated using a sample of term structures $R = \left\{R_T^{(i)}\right\}_{i=1,\ldots,n}$. The drift parameters ϑ_t are estimated to match the term structure while the mean-reversion and volatility parameters are estimated to match the mean-reversion and volatility in the whole term structure. Therefore, the estimation approach is a mix of calibration (when drift parameters are calibrated to match the term structure) and regression analysis that matches the volatility and mean reversion to historically observed volatility and mean reversion of the term structure.

The Ho-Lee term structure model is described by the following equation

$$R_{t,T} = r_t - \frac{1}{6}\sigma^2 T^2 + \frac{1}{T}\int_0^T \vartheta_s \times (T-s)ds$$

where

$$dr_t = \vartheta \times dt + \sigma \times dW_t$$

The equations show that the term structure in the Ho-Lee model shift in parallel over time:

$$dR_{t,T} = dr_t = \sigma \times dW_t$$

for all T.

The approach is implemented as follows for the Ho-Lee model.

- For the fixed σ and each observed term structure $R_T^{(i)}$, the following series is constructed

$$y_T^{(i)} = TR_T^{(i)} - Tr_0 + \frac{1}{6}\sigma^2 T^3$$

- Parameters ϑ_t are estimated as follows. For a piece-wise constant function ϑ_t, the formula $\int_0^T \vartheta_s \times (T-s)ds$ can be represented as follows

$$\int_0^T \vartheta_s \times (T-s)ds =$$

- The difference between

Parameter Sensitivity

The section studies how sensitive is parameter calibration / estimation to the noise in the underlying yield sample data.

Hull-White Model of Interest Rates

Hull-White model is applied to risk-free bonds which price depends on the stochastic movement of the risk-free rates. The exposure to bond default risk is not considered either because the bond is issued by a highly-rated entity or because it is collateralized and can be assumed to be risk-free. The stochastic structure is modelled using a diffusion process of short-term interest rates.

Hull-White model of interest rates

$$dr_t = (\theta_t - \alpha r_t)dt + \sigma_t dW_t \quad \text{(R.1)}$$

Construction of tree approximation of the process

Stage A: construct the discrete tree approximation for the following process

$$dR_t = -\alpha R_t dt + \sigma_t dW_t$$

The process eliminates the drift parameter θ_t and models the process that is symmetric around $R^* = 0$.

The symmetric tree is modelled as a trinomial tree with the step

$$\Delta R = \sigma\sqrt{3\Delta t} \quad \text{(R.2)}$$

The tree region is divided in three separate regions:

Tree interior: Symmetric tree branching

The number of states in period t is $N_t = 2 \times t + 1$. The transition probability mapping is set using the following local function:

$$R_t \Rightarrow \begin{cases} R_t + \Delta R; & p_u \\ R_t; & p_m \\ R_t - \Delta R; & p_d \end{cases} \quad \text{(R.3)}$$

The probabilities (p_u, p_m, p_d) are constructed from the following conditions:

i. **Expectation** of the conditional transition probabilities: $E[dR] = -\alpha R\Delta t$ or, in the discrete form, $p_u\Delta R - p_d\Delta R = -\alpha R_t\Delta t$;
ii. **Variance** of the conditional transition probabilities: $E[dR^2] = Var[dR] + E^2[dR] = \sigma^2\Delta t + (\alpha R\Delta t)^2$ or, in the discrete form, $p_u(\Delta R)^2 + p_d(\Delta R)^2 = \sigma^2\Delta t + (\alpha R\Delta t)^2$;
iii. $p_u + p_m + p_d = 1$;

The system of equations for (p_u, p_d) can be presented as

$$\begin{cases} (p_u - p_d) \times (\Delta R)^2 = -\alpha R_t \Delta R \Delta t \\ (p_u + p_d) \times (\Delta R)^2 = \sigma^2\Delta t + (\alpha R\Delta t)^2 \end{cases}$$

Note that $\frac{\Delta t}{(\Delta R)^2} = \frac{1}{3\sigma^2}$, and therefore the system can be represented as

$$\begin{cases} p_u - p_d = -\dfrac{\alpha R_t \Delta R}{3\sigma^2} \\ p_u + p_d = \dfrac{1}{3} + \dfrac{(\alpha R)^2 \times \Delta t}{3\sigma^2} \end{cases}$$

Solving the system, we obtain

$$\begin{cases} p_u = \dfrac{1}{6} + \dfrac{\alpha R}{6\sigma^2}[\alpha R \times \Delta t - \Delta R] \\ p_d = \dfrac{1}{6} + \dfrac{\alpha R}{6\sigma^2}[\alpha R \times \Delta t + \Delta R] \end{cases}$$

and

$$p_m = 1 - p_u - p_d = \frac{2}{3} - \frac{\alpha R}{3\sigma^2} \times \alpha R \times \Delta t$$

In the case of the interest process with zero mean reversion ($\alpha = 0$), the formulas are simplified to

$$\begin{cases} p_u = p_d = \dfrac{1}{6} \\ p_m = \dfrac{2}{3} \end{cases}$$

The transition probabilities are described in this case by the following simplified formula:

$$R_t \Rightarrow \begin{cases} R_t + \sigma\sqrt{3\Delta t}; & \dfrac{1}{6} \\ R_t; & \dfrac{2}{3} \\ R_t - \sigma\sqrt{3\Delta t}; & \dfrac{1}{6} \end{cases} \qquad \text{(R.4)}$$

Tree top boundary: Downward tree branching

Tree bottom boundary: Upward tree branching

Interest Rate Option Calculations

Interest rate options are derivative contracts that depend on the state of the modelled stochastic interest rate (or, respectively, on the bond price). The options are estimated by modelling the value of the bond conditional on different actions. For example, in the case of put / call options the bond value is estimated as maximum / minimum price of the bond conditional on the selected option to exercise the option. The estimation process is performed using the following steps:

1. Construct the tree model of the stochastic interest rates;
2. Estimate the option-free bond price running backward recursion and default no-exercise action;
3. Estimate the bond price in the presence of the option by maximizing / minimizing the price of the bond with respect to the set of actions;
4. Calculate the value of the option as the difference of the bond price with and without the option;

The list of standard interest rate options includes:

i. **Prepayment (call) option** that gives the borrower the right (but not the obligation) to repay the bond principal and accrued interest at no penalty or make-whole provisions. The option is exercised so that to minimize the price of the bond to the borrower. The bond is exercised at low interest rates so that it is the benefit to the borrower to exercise the call option and refinance the debt obligations at lower interest rates;
ii. **Mandatory repayment (put)** option that gives the lender the right (but not the obligation) to request the repayment of the bond principal and accrued interest at no penalty or make-whole provisions. The option is exercised so that to maximize the price of the bond to the lender. The bond is exercised at high interest rates so that it is the benefit to the lender to exercise the put option and reinvest the funds at higher interest rates;
iii. **Forward** option that sets a given fixed (exercise) price of the bond at a given future date. The exercise price is typically set at the effective date so that the value of the future option is zero. The explicit value of the forward option can be calculated using Black formula (discussed in the examples below);
iv. **Interest deferral** option that gives the right to the borrower to defer interest payments until a given date in the future. With increasing interest rates exercising the interest deferral optio presents a benefit to the borrower.

The above options can be modelled under more complex conditions such as

a. Presence of the **penalty provisions** that is often in the form of the non-par price schedule specified for the option. In the case of the call option, lots of agreements set the above-par (decreasing to par) price schedule at which the call option can be repaid;
b. Presence of the **make-whole provision** that requires to repay the debt (for the call option) using a low discount rate (treasury rate plus a small spread). The option is exercised only in the event of very significant drop in the interest rates;
c. Presence of the **mandatory amortization provision** that reduces the value of the call option since the call option is applied to smaller notional amount over time;

Examples

A call option is modelled as follows.

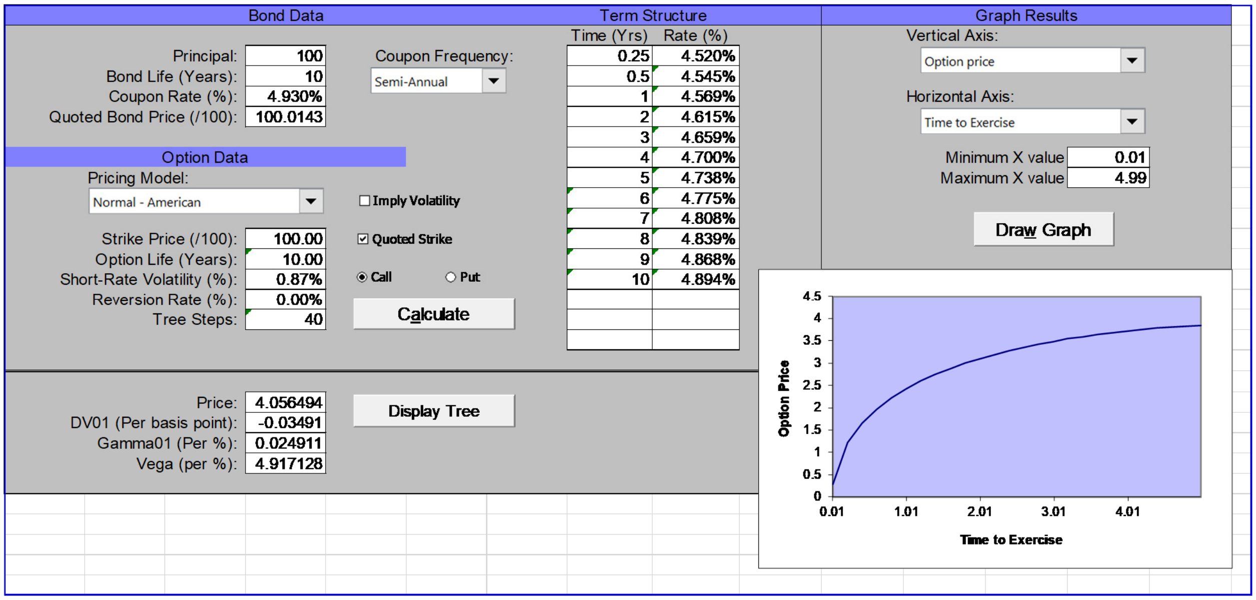
1. **Initial state**: $r_0 = c$; the initial yield rate is set equal to the coupon rate. It is assumed that the call option is valued at the initial date when the bond is valued at par;
2. **Frequency of coupon payments**: f. The parameter can take values $f = \{1,2,4\}$;
3. **Volatility**: volatility σ is measured using historical short-term yield data. Under the assumption of the Hull-White model described above and zero mean-reversion parameter, the interest rate process is described as $dr_t = \theta dt + \sigma dW_t$ so that parameter σ can be estimated based on the standard deviation of the daily dr_t series which is normalized then by the number of business days in the years (multiplied by $\sqrt{250}$ where it is assumed that a regular year has 250 business days). As a more accurate measure, the parameter σ can be estimated based on the standard deviation of $\frac{dr_i}{\sqrt{dt_i}}$ sample where $dt_i = \frac{\#\ of\ days\ between\ two\ periods}{\#\ of\ days\ in\ the\ year}$. The number of days in the year is generally set by default to 365 or 365.25 (to take into account the leap years);
4. **Number of periods**: $T = 4 \times T^{yr}$ where T^{yr} is the maturity of the bond in years. Each step of the tree corresponds to a three-months period;
5. **Set of actions**: $\mathcal{A} = \{0,1\}$ where element $a = 0$ corresponds to no-exercise and $a = 1$ to exercise action.
6. **Set of states**: $\mathcal{S} = \{r_0 - i\Delta R, r_0 + i\Delta R, i = 0, \dots, T\}$ & $\{r^*\}$ where r^* is the exercise set which is selected outside (typically above) the regular states of the interest rate tree.
7. **Discount rate**: the local discount rate is estimated as $D_t(x) = \begin{cases} \frac{1}{(1+r_t)^{\frac{1}{4}}} & if\ r_t \neq r^* \\ 0 & if\ r_t = r^* \end{cases}$.
8. **Terminal function**: in the absence of the call option, $f_T(x) = 100 \times \left(1 + \frac{c}{f}\right)$. In the presence of the call option, $f_T(x) = 100$;
9. **Objective function**: $g_t(x) = \begin{cases} c_t & if\ r_t \neq r^* \\ C_t(x) & if\ r_t = r^* \end{cases}$, where c_t is the coupon payment in period t and C_t is a given cost of exercising the call option in period t and state x.

DerivaGem Tool

DerivaGem (current version 2.01) is a freeware implemented as an excel file to calculate the interest rate options including call / put options. The official website for the tool is http://www-2.rotman.utoronto.ca/~hull/software/. The screen for the bond option pricing is displayed below. The following parameters are set to estimate the call / put option price:

1. Principal amount (value of the call option is calculated relative to the principal amount. Therefore, if the principal amount is set equal to 100, then the call option is calculated as a percentage of the principal amount);
2. Maturity term of the bond (in years);
3. Coupon rate is set equal to the bond coupon rate;
4. **Strike price** (**redemption price** of the bond). The strike price is set equal to the principal amount if there is no penalty for the option prepayment. DerivaGem tool can model only constant penalty over time. If penalty is $x\%$, then the strike price is set at $K = 100 \times (1 + x\%)$;
5. Option life (in years). Option life typically equals to the maturity term of the bond minus notice period of the option minus one day;
6. Short-rate **volatility** of the market yields. The market yield rates are used to estimate the short-rate discount factors used in bond valuation. The volatility is estimated based on historical behavior of the short-rate (3-months) yield series. The credit risk of the yield series is selected to match the credit rating of the reference bond to take into account the credit risk exposure in the option transaction (if the bond defaults, then the payouts in the options are set to zero). Alternatively, the volatility can be estimated using treasury rates (assuming that valuation is performed using treasury rates) but the default state of the bond and the varying probability of default must be modeled explicitly. For Hull-White (extended Vasicek) model volatility is estimated as $\hat{\sigma} = \sqrt{250} \times stdev\,[dr_t]$;
7. **Mean-reversion rate**. The default mean-reversion rate was assumed to be equal to zero;
8. Tree steps. Generally is set to four times the maturity term. Factor four corresponds to a 3-month length of each discrete time period in the interest rate tree;
9. **Term structure**. Each model of the short-rate (Hull-White extended Vasicek or extended CIR) produces a term-structure specific to the model. For example, the term structure for the Hull-White extended Vasicek model, assuming zero mean-reversion rate, is equal to $R_T = r_0 + \frac{1}{2}\vartheta T - \frac{1}{6}\sigma^2 T^2$. The term structure must be set in the corresponding cells. To estimate the term structure, the drift parameter ϑ and the initial value r_0 must be set. The drift parameter ϑ effectively determines the slope of the term structure. The parameter is calibrated from the term structure estimated as of the valuation date: $\vartheta = \frac{2}{T}(R_T - r_0) + \frac{1}{3}\sigma^2 T$. Parameter r_0 is set to match the bond price to the quoted bond price as discussed below;
10. **Quoted Bond Price**. If the valuation is performed as of the bond issue date, the quoted bond price is generally set to par (100) value. The par value must be consistent with the term structure parameters of the model. We calibrate the parameter r_0 so that the quoted bond price equals to bond par value.
11. Frequency of the coupon payments is set to semi-annual to match the coupon frequency of standard USD bonds. Generally the parameter must match the frequency of the bond coupon payments;

12. The interest rate model is set to “`Normal – American`”. The model is described by equations (3.1) – (3.4).



DerivaGem Tool: Comments

Applicability of the DerivaGem tool depends on the following implicit assumptions:

i. In the Hull-White extended Vasicek model market yield rates can move into the region of negative values. Under the assumptions above, the market yield rates are symmetric with respect to the initial yield rate and are not bounded from above or below. Symmetric tree branching is applied in each period and at each state;

ii. The valuation tool does not take into account the default probability on the bond obligations;

DerivaGem: Example

Let’s consider the following example. The call option is priced using the following parameters.

1. Principal and strike price are set to 100;
2. Bond maturity term and option life are set to 2.5 years; the number of tree steps is set to $2.5 \times 4 = 10$ (10 tree steps is the maximum number of tree steps that can be displayed by the tool);
3. Coupon rate is set equal to the yield rate and is set to 3%;
4. Volatility is set to 0.5% (historical volatility rate estimated for the interest rate model described by equation (3.1) typically ranges between 0.1% and 1.0%); mean-reversion rate is set to zero;
5. Coupon frequency is set to semi-annual and the model is set to “Normal – American”;

Under the parameters, the market yield rate in the last period ranged between -1.32% and 7.34% (approximately 4.33% deviation from the initial 3% yield rate). The call option price in period $t = 0$ was estimated at $C_0 = 1.7759$; The call option in the last period was estimated equal to the coupon payment: $C_T = 0.5 \times 3\% = 1.5\%$.

The example was replicated as the following optimization problem on the stochastic interest rate tree.

1. Discount rate function: $D_t(x) = \begin{cases} \frac{1}{(1+r_t)^{\frac{1}{4}}} & if\ r_t \neq r^* \\ 0 & if\ r_t = r^* \end{cases};$
2. Terminal function: $f_T(x) = \begin{cases} 100\% \times \left(1 + \frac{3\%}{2}\right) & if\ r_t \neq r^* \\ 100\% & if\ r_t = r^* \end{cases};$
3. Objective function: $g_t(x) = \begin{cases} 0 & if\ r_t \neq r^*\ and\ t\ is\ odd \\ \frac{3\%}{2} & if\ r_t \neq r^*\ and\ t\ is\ even\ (and\ positive) \\ 100\% & if\ r_t = r^* \end{cases};$

Note that even positive values of t correspond to coupon payment dates. The state r^* was set at $r^* = 100\%$.

The borrower exercises the option at low market yield rates when the price of the bond is above the par and therefore exceeds the value of exercising the call option. From the borrower's perspective the optimal action to exercise the call option is selected to minimize the price of the bond. Therefore, the optimization problem is solved with the terminal and objective functions set with negative signs.