Estimation of yield rate term structure using Nelson-Siegel family of parametrized curves

The term structure of interest rates is the relationship between interest rates or bond yields and different terms or maturities. The term structure of interest rates is also known as a yield curve. Yield curves are usually upward sloping asymptotically: the longer the maturity, the higher the yield, with diminishing marginal increases. There are two common explanations for upward sloping yield curves. First, it may be that the market is anticipating a rise in the risk-free rate. Another explanation is that longer maturities entail greater risks for the investor and therefore investors demand a higher risk premium to invest in long-term bond instruments.

Yield curves are built from either prices available in the bond market or the money market. The yield curves built from the bond market use prices only from a specific class of bonds (for instance US Treasury bonds or US corporate bonds with a given consensus credit rating). A standard approach to estimate yield curves is to approximate the bond yield data using parameterised curves (such as splines, the Nelson-Siegel family, the Svensson family, or the Cairns restricted-exponential family of curves).

The yield curve estimation based on Nelson-Siegel family of parametrized curves is a traditional approach to term structure modeling in the financial industry.[1] Nelson-Siegel term structure model ("NSM") is a three-factor model, in which the term structure is estimated based on a sample of bond yield data $\{y_i\}$ and is described by the following equation:

$$y_i = f_1 + f_2\left(\frac{1 - e^{-\alpha\tau_i}}{\alpha\tau_i}\right) + f_3\left(\frac{1 - e^{-\alpha\tau_i}}{\alpha\tau_i} - e^{-\alpha\tau_i}\right) + \varepsilon_i \qquad \text{(E1)}$$

where y_i is the yield rate on bond i that has remaining maturity term τ_i, f_1, f_2, f_3 and α are unknown parameters of the Nelson-Siegel family of parametrized curves, and ε_i is the error term.

As an illustrative example, the bond yield data $\{y_i\}$ was estimated by running a search for the US$ denominated bonds issued in US and Canada by companies operating in broad industrial sector that have a Caa2/CCC credit rating reported consistently by both Moody's and S&P rating agencies. The search was performed using Bloomberg database and the search identified 28 bond transactions that matched the above criteria.

The unknown parameters f_1, f_2, f_3 are referred to as model factors and are interpreted as follows: (i) f_1 is the term structure *level* factor; (ii) f_2 is the term structure *slope* factor; and (iii) f_3 is the term structure *curvature* factor.[2] Functions $\Lambda_{1,\tau} = 1$, $\Lambda_{2,\tau} = \frac{1-e^{-\alpha\tau}}{\alpha\tau}$, and $\Lambda_{3,\tau} = \frac{1-e^{-\alpha\tau}}{\alpha\tau} - e^{-\alpha\tau}$ are referred to as factor loadings.

The unknown parameters are estimated as follows:

1. Select parameter α of the Nelson-Siegel curve family. There are several methods to estimate the α parameter that have been recommended in the literature:[3]
   a. Diebold-Li: $\alpha = \frac{1}{1.37}$;
   b. Fabozzi: $\alpha = \frac{1}{3.0}$;
   c. Nelson-Siegel: estimate parameter α, which maximizes the R^2 goodness-of-fit measurement of the regression equation (E1).

   In practice the three options typically produce similar results.
2. Run ordinary least square (OLS) method to estimate parameters f_1, f_2 and f_3 of the linear regression equation (E1).

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[1] Bolder D and D Stréliski, "Yield Curve Modelling at the Bank of Canada", Technical Report No. 84, Bank of Canada, February 1999.

[2] Parameter f_1 can be interpreted as the term structure asymptotic long-term yield rate and parameter $-f_2$ can be interpreted as the term premium (defied as the difference between the long-term and the short-term yield rates).

[3] Jan Annaert, Anouk G.P. Claes, Marc J.K. De Ceuster, Hairui Zhang, "Estimating the Yield Curve Using the Nelson-Siegel Model: A Ridge Regression Approach", Universiteit Antwerpen, Belgium, Louvain School of Management, Belgium

The yields on CCC rated bonds are typically highly volatile and some yield rates in the identified sample can deviate significantly from the estimated term structure. The yield rates with significant estimated error term ε_i are interpreted as yield rate outliers. In many cases, the outlier yield rates are typically significantly higher than the estimated term structure yields. The increase in the outlier yield rates can be explained by a future downgrade in the related bond credit rating (and in some cases default on the bond), removal of the bond credit rating or other factors. These factors however cannot be applied to eliminate the bonds from the sample as they are observed post factum after the valuation date of the yield term structure. The market bond prices typically respond much faster to the negative news and the rating agencies update the bond credit ratings post factum when the yield rates on the bonds are already significantly higher than the market yields on the bonds with the same credit rating profile.

Presence of the outliers in the sample results in the biased and unstable estimation of the yield term structure. Therefore we apply the following procedure to remove the outliers from the final sample.

1. Estimate parameters f_1, f_2, f_3 and α of the Nelson-Siegel family of parametrized curves as described above;
2. Identify the bond i with the largest absolute value of the error term ε_i (which measures the deviation of the yield rate y_i from the estimated term structure);
3. Remove bond i from the sample and repeat from step 1 until all error terms ε_i are less in absolute value than a certain selected threshold value.

To summarize, the term structure of the yield rates was estimated as follows:

1. We searched for the US$ denominated bonds issued in US and Canada by companies operating in broad industrial sector that have a Caa2/CCC credit rating reported consistently by both Moody's and S&P rating agencies. Estimate the yield rates $\{y_i\}$ as of the valuation date of the analysis;
2. We removed the outliers from the sample as described above;
3. We estimated the yield term structure using Nelson-Siegel family of parametrized curves and a final sample of bond yield rates $\{y_i\}$.

The estimated term structure for the CCC rated industrial yield rates is presented in the exhibit below.

Term structure of CCC rated yield rates

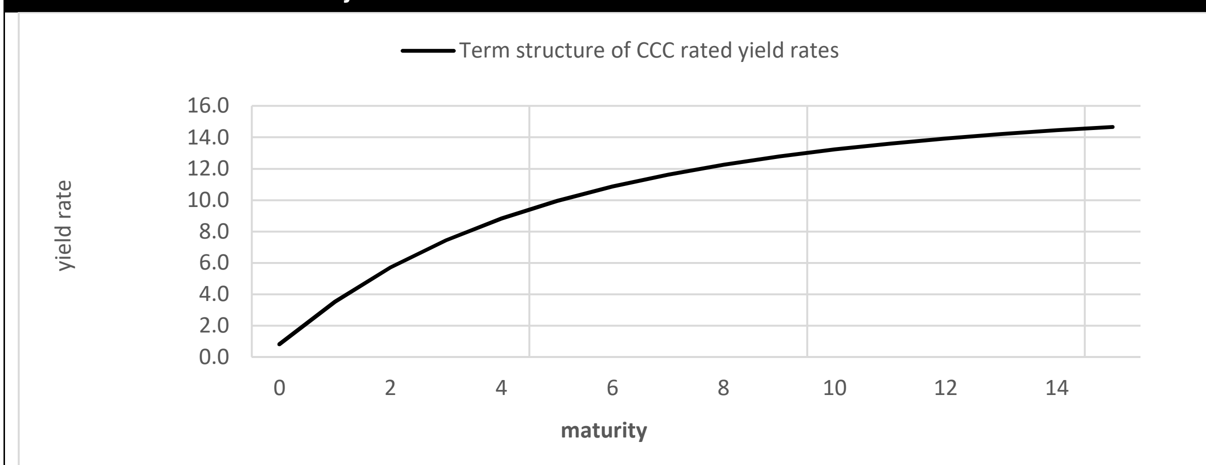
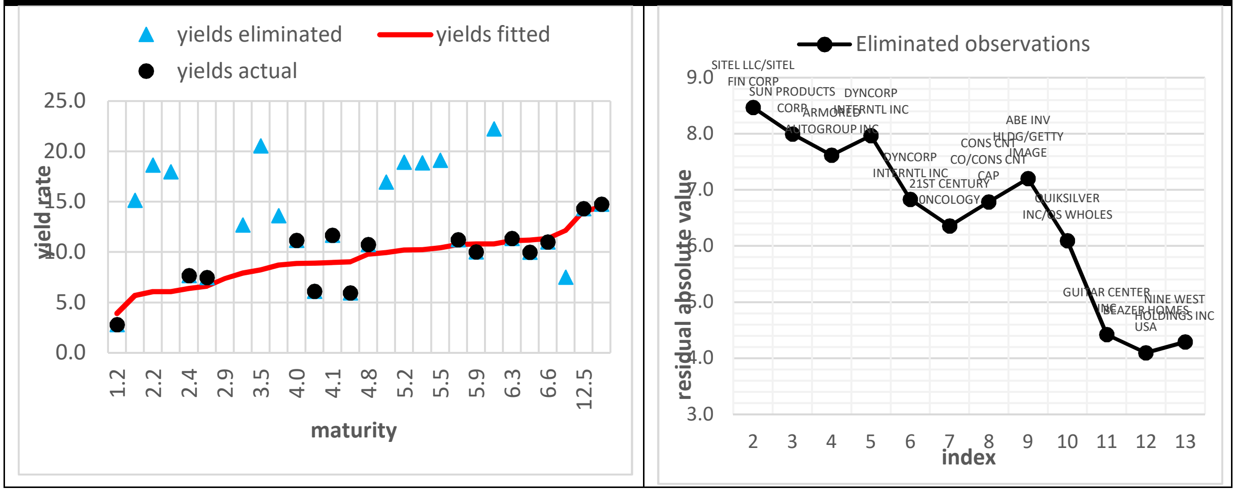


Exhibit below shows the yield rates retained in the final sample, yield rates retained in the final sample, and the yields fitted using Nelson-Siegel family of parametrized curves (left panel). The exhibit shows also the outliers eliminated from the final sample (right panel). In total 15 bonds were retained in the final sample and 13 bonds were eliminated.

Term structure of US$ CCC rated yield rates (left panel) and yield rate outliers eliminated from the final sample (right panel)



The final sample of CCC rated bonds used to construct the term structure is summarized in the exhibit below.

Final sample of CCC rated bonds used to construct the term structure of CCC rated yield rates

#	Ticker	Issuer name	Industry sector	Maturity date	Remaining tenor	Yield rate
1	EK417798 Corp	BWAY HOLDING CO	Packaging & Containers	15-Aug-21	6.3	11.3560
2	EI724825 Corp	BEAZER HOMES USA	Home Builders	15-May-19	4.1	6.1110
3	EF656286 Corp	BEAZER HOMES USA	Home Builders	15-Jun-16	1.2	2.8000
4	EK054663 Corp	BEAZER HOMES USA	Home Builders	15-Sep-21	6.4	9.9730
5	EK354818 Corp	BEAZER HOMES USA	Home Builders	15-Jun-19	4.2	5.9540
6	EK056998 Corp	BIOSCRIP INC	Pharmaceuticals	15-Feb-21	5.8	11.2280
7	EJ587873 Corp	CERIDIAN HCM HOLDING INC	Commercial Services	15-Mar-21	5.9	10.0220
8	EI810032 Corp	DJO FIN LLC/DJO FIN CORP	Healthcare-Products	15-Oct-17	2.5	7.4700
9	EC117098 Corp	MCCLATCHY CO	Media	15-Mar-29	13.9	14.7560
10	DD112817 Corp	MCCLATCHY CO	Media	1-Nov-27	12.5	14.2960
11	EF062853 Corp	MCCLATCHY CO	Media	1-Sep-17	2.4	7.6680
12	EK227300 Corp	NINE WEST HOLDINGS INC	Apparel	15-Mar-19	3.9	13.6100
13	EK056164 Corp	NORANDA ALUMINUM ACQUISI	Mining	1-Jun-19	4.1	11.6770
14	EK048683 Corp	URBAN ONE INC	Media	15-Feb-20	4.8	10.7360
15	EK586550 Corp	TIBCO SOFTWARE INC	Internet	1-Dec-21	6.6	10.9900