INTEREST RATE OPTIONS

A TECHNICAL GUIDE

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List of Abbreviations

The following abbreviations and symbols are used in this guide:

ac.finance.SRM	Java/Excel – based interest rate option valuation tool developed as part of this guide
ARN	Adjustable rate note
Bloomberg	Bloomberg Professional Service
bps	Basis points
CBOE	Chicago Board Options Exchange
CIR	Cox-Ingersoll-Ross
DerivaGem	Hull-White interest rate option valuation tool
EWMA	Exponentially weighted moving average
GARCH	Generalized auto-regressive conditional heteroskedasticity
GARP	Global association of risk professionals
HP filter	Hodrick-Prescott filter
HW	Hull-White
Moody’s	Moody's Investors Services, Inc.
OECD Guidelines	“BEPS Actions 8 – 10, Financial Transactions”, a draft published in July – September 2018 for the purposes of public discussion
Reuters	Reuters news agency, a subsidiary of Thomson Reuters Corp.
SA	Semi-annual
SRM	Short rate model
SRVIX	Swap Rate Volatility Index

Section 1 Introduction

The notes summarize the approaches applied to valuation interest rate options including (i) prepayment (call) option and (ii) pay-on-demand (put) option.

A prepayment option is a standard term in a bond transaction. In a typical bond sample more than half of the bonds are often callable bonds.[1] Presence of a put option is a much less typical term of a bond transaction. However, putable bonds are also periodically observed in a sample of bond transactions.

In an intercompany loan transaction, the prepayment (or pay-on-demand) option is typically included for two reasons:

1. The call option is included to maximize the interest rate on the loan, which is consistent with the market interest rates. (In the case of a pay-on-demand option the presence of a put option decreases the interest rate on the loan);
2. The parent group of the borrower wants to have an option to unwind the debt structure if deemed necessary. Presence of the call (put) option provides such an option for the parent group.

Presence of the call (put) option may result in a transfer pricing risk that the borrower may have an incentive to refinance the loan at some time in the future. Therefore, the interest rates need to be monitored on a regular basis to ensure that the borrower does not have an incentive to exercise the call option. An example of debt refinancing is provided in Appendix F.4. Presence of a penalty structure in a prepayment option partially mitigates the risk.

The objective of the interest rate option pricing tool, which is developed as a part of this guide, is to (i) adjust the yield rates on comparable callable (putable) bonds and to (ii) estimate the prepayment (pay-on-demand) premium (discount) for the tested transaction.

1.1 Interest rate options

Bond agreements often include different options such as an option for the borrower to repay the bond early prior to the bond maturity date or the option for the lender to demand and early bond repayment. Early bond repayment allows the borrower to take advantage of lower market interest rate and refinance the bond at lower cost. Similarly, a pay-on-demand option allows the lender to take advantage of high market interest rates and reinvest the funds in a higher yield instruments with the same credit risk.

To exercise the prepayment (pay-on-demand) option, the borrower (lender) is typically required to provide a notice period to the counterparty prior to exercising the option. Exercising the option may involve a penalty either in the form of a make-whole provision or a premium, which generally depends on the remaining maturity term of the bond.

The primary risk that is modelled to evaluate the option value is the market interest rate risk. Decrease in market interest rates increases the chances of the bond prepayment. Similarly increase in market in interest rates increases the chances that the lender will demand the bond prepayment. In practice pay-on-demand option is rarely included in the bond agreement and when included the option can be typically exercised in

[1] Bloomberg typically classifies a callable bond with the make-whole provision that is effective until the bond maturity date as a bullet bond. We also treat these bonds as effectively non-callable bonds. Many callable bonds also have make-whole termination date which is a few months prior to the bond maturity dates. These bonds are also treated as effectively non-callable.

a few discrete dates, where the first exercise date is set at least five years after the bond issue date. The prepayment option is a much more typical feature of a bond agreement. However, in most cases the prepayment option has a penalty structure in the form of both make-whole provision and prepayment premium penalty structure. A typical penalty structure is described in the Appendix B.

Valuation of interest rate options involves the following steps:

- Model a stochastic process of interest rates. The process is typically described by an interest rate tree and the probability distribution of the interest rates over the tree states;
- Estimate the borrower (lender) prepayment (pay-on-demand) decision, which specifies in which interest rate tree state the prepayment (pay-on-demand) is exercised. The option is exercised whenever the value of exercising the option (which takes into account the option penalty structure) exceeds the value of keeping the bond;
- Estimate the value of the prepayment (pay-on-demand) option as the difference between the value of the bond with the option and the value of the bullet (option-free) bond.

Multiple factors may affect the stochastic changes in the market interest rates. For example, a Nelson-Seidel modelling approach[2] describes interest rate process as a three-factor model, where the three factors (level, slope, and curvature) represent the geometric shape of the interest rate term structure. Interest rate options are typically modelled using one-factor models. The models use one-factor process to describe short-term interest rates. The interest rates with other maturity terms are derived analytically from the one-factor short-term rates. Under the approach, the movements in the interest rates with different maturity terms are affected by a single stochastic factor and therefore the movements are strongly correlated.

A special case of the one-factor interest rate models are interest rate models with affine term structure. The interest rates have affine term structure whenever the prices of zero-coupon bonds are described by the following equation

$$P_T = A(t,T) \times e^{-B(t,T)\times r_0}$$

where $A(t,T)$ and $B(t,T)$ are some arbitrary functions. The respective interest rate term structure is described by the following equation

$$R_t = -\ln P_T = -\ln A(t,T) + r_0 \times B(t,T)$$

Under the affine term structure modelling approach, the interest rates with arbitrary maturity term T are a linear function of the short-term interest rates r_0. The analytical solution of the interest rate term structure allows (i) to derive model parameters and (ii) to validate the results of the interest rate option valuation analysis. The details of model parameter estimation and model output validation are provided in the sections below.

This guide describes four alternative affine term structure models. Two parametric models include Vasicek model and Cox-Ingersoll-Ross (**CIR**) model. The parametric models are described by three numeric parameters: drift, volatility, and mean-reversion. Two non-parametric models include Hull-White (extended Vasicek) and Hull-White (extended CIR) models. The volatility and mean-reversion parameters in the non-parametric models are still assumed to be numeric. The drift parameter in the non-parametric models are assumed to be functions of maturity term t.

[2] The approach is described in more detail in a separate “Interest Rate Benchmarking” guide.

1.2 Terminology

The following terminology is used throughout these notes.

- ► **Bullet** bond. A bullet bond is a bond where a payment of the entire principal of the bond, and sometimes the principal and interest, is due at the bond maturity term. Effectively a bullet bond is a bond with no prepayment and pay-on-demand options.
- ► **Call** option. An early prepayment option providing a borrower with a right to prepay outstanding principal and accrued interest amounts.
- ► **Put** option. A pay-on-demand option allowing a lender to demand an early repayment of outstanding principal and accrued interest prior to a maturity date.
- ► **Make-whole provision**. Under a make whole clause the borrower has to value the cash flows beyond the date of the bond early call/redemption date. The valuation is performed using a low discount rate (for example, the discount rate is estimated as the yield on government bond plus a small spread). The estimated bond full value (which includes future bond cash flows) is compensated to the lender if the call option is exercised. The purpose of the make-whole provision is to provide a strong form of protection for lenders/investors in securities, designed to mitigate the adverse effects of call risk for investors. The consequence of a make whole clause for the investor is that they can re-invest the redemption monies in government stock, thus preserving their originally expected cash inflows at lower risk.

  Potentially it makes prohibitively expensive for the borrower to take an early redemption under the make-whole termination provision. In practice the make-whole provision often has a termination date. If the make-whole termination date matches the maturity date, then Bloomberg typically refers to the bond as a bullet bond. In this guide we assume that exercising the call option has an infinite cost prior to the make-whole termination date (so that the call option is never exercised prior to the make-whole termination date) and the prepayment penalty is determined by the bond penalty structure after the make-whole termination date.
- ► **Notice period**. The notice period is the time period between the receipt of the notice that the option will be exercised and the actual option exercise date. It is assumed in this guide that the notice provided by the borrower is a commitment that the option will be exercised at the end of the notice period.
- ► **Soft call**. Soft call protection requires the payment of a premium to the investor, on any early redemption of a callable bond by the borrower. At early redemption the premium becomes payable, together with principal and outstanding interest at the call/redemption date.

  Soft call is an alternative to the make-whole provision (hard call). Soft call is a weak form of protection for lenders/investors in securities, designed to mitigate the adverse effects of call risk for investors. It sometimes applies only for an early part - for example just the first year - of the life of a security (the security becoming freely callable after that initial period of soft call protection).
- ► **Arbitrage-free interest rate models** are the models (such as Hull-White extended Vasicek and extended CIR models), in which the term structure of yield rates is matched exactly.

1.3 Valuation summary

The notes summarize the steps performed in a selection and option estimation for a specific family of interest rate models. The discussion is provided for Hull-White (extended Vasicek) and Hull-White (extended CIR) models (however the list can be extended if necessary). The steps are summarized below and are described in the following sections.

1. Select the **family of interest rate models**
   - ► Vasicek;
   - ► Hull-White (extended Vasicek) ;
   - ► Cox-Ingersoll-Ross (CIR);
   - ► Hull-White (extended CIR);
2. Estimate **parameters of the model**. Each of the above two families depends on the following three parameters: (i) volatility; (ii) drift; (iii) mean reversions, and (iv) coupon value. Parameters are currently assumed to be estimated as follows:
   - ► **Volatility** is estimated based on a historical sample of short-term rates;
   - ► **Drift** is estimated based on the latest term structure of yield rates;
   - ► **Mean-reversion** parameter is assumed to be zero.
   - ► **Coupon rate** value is calibrated so that the bond bullet value is equal to par.
3. Review the adjusted parameters and adjust manually if necessary. For the parametric models all three parameters (drift, volatility, and mean-reversion) can be manually overridden. For the non-parametric models only numeric parameters (volatility and mean-reversion) can be manually overridden.
4. Estimate interest rate options.
5. Validate the results of option estimation analysis.
   - ► Derive the formulas of zero-coupon prices and respective yield term structure. Zero-coupon prices are used as proxies for the discount factors in option calculation. Yield term structure is used to derive the drift parameter in the step above. Zero coupon prices are also used to derive theoretical bond price and compare it against the estimated numerical bond prices;
   - ► Derive the formulas for the terminal interest rate distribution parameters. The parameters are used in the test against the calculated numerical distribution parameters.
   - ► Derive the formulas for the implied model parameters based on the estimated terminal distribution parameters. Compare the implied model parameters with the actual parameters to assess how material is the error produced by the model discrete approximation implementation.
   - ► Validate the results directly using **DerivaGem** as an alternative option valuation tool.

Section 2 Interest Rate Model Families

General form of interest rate process can be represented as follows:

(2.1)
$$dr_t = \mu(t, r_t) \times dt + \sigma(t, r_t) \times dW_t$$

where different models are summarized in the exhibit below. The Vasicek and CIR models are parametric models described by three parameters: drift, volatility, and mean-reversion. For simplicity, the mean – reversion parameter is assumed to be zero. The volatility parameter is estimated based on historical sample of short-term rates. The drift parameter is estimated to approximate the term structure of the yield rates.

The guide describes four special types of the affine structure interest rate models. Vasicek and CIR models are parametric models, which are described by drift, volatility, and mean-reversion parameters (denoted respectively as ϑ, α, and σ). Mean-reversion parameter is assumed to be zero. Volatility parameter is estimated based on historical sample of short-term rates. Drift parameter is estimated so that to match approximately the interest rate term structure.

2.1 Model specification

Alternative interest rate model specifications are summarized in the Exhibit 2.1 below.[3]

Exhibit 2.1 Summary of interest rate model specifications

Model name	Drift term	Diffusion term	Comment
Vasicek (1977)[4]	$\mu_t = \vartheta - \alpha r_t$	σ	**Approximate** matching of the term structure (using constant slope parameter); **unbounded** interest rate; **constant** interest rate volatility
Hull-White (extended Vasicek, 1990)[5]	$\mu_t = \vartheta_t - \alpha_t r_t$	σ_t	**Exact** matching of the term structure (using constant slope parameter); **unbounded** interest rate; **constant** interest rate volatility
Cox – Ingersoll – Ross (CIR, 1985)	$\mu_t = \vartheta - \alpha r_t$	$\sigma_t r_t^{\frac{1}{2}}$	**Approximate** matching of the term structure (using constant slope parameter); **bounded** interest rate (with zero); interest rate volatility **increases** with interest rates
Hull-White (extended CIR, 1990)	$\mu_t = \vartheta_t - \alpha_t r_t$	$\sigma_t r_t^{\frac{1}{2}}$	**Exact** matching of the term structure (using constant slope parameter); **bounded** interest rate (with zero); interest rate volatility **increases** with interest rates

[3] Other affine term structure model specifications have also been studied in the financial literature including the following models: Dothan (1978), Rendleman – Bartter, Courtadon, Constant Elasticity of Variance (CEV), Marsh-Rosenfeld (1983), Exponential Vasicek (EV), Black-Derman-Toy (1990), Black-Karazinski (1991), and other. However, these models are outside the scope of this guide.

[4] The ϑ parameter in the Vasicek model is assumed to be of the form $\vartheta = \alpha b$, where b is the long-term steady state of the interest rates. In this form the random walk model ($\alpha = 0$) cannot be modelled as a special case of Vasicek model. We present the Vasicek model in the format consistent with the Hull-White (extended Vasicek) format.

[5] Hull-White describe formally the drift parameter in the model as follows: $\mu_t = \vartheta_t + \alpha(b - r)$. We assume that $b = 0$ or equivalently that the ϑ_t term represents the $\vartheta_t + \alpha b$ term in the original Hull-White model. Parameters α_t and σ_t in the Hull-White (extended Vasicek) and Hull-White (extended CIR) models are assumed to be constant through these notes.

The Hull-White (extended Vasicek0 and Hull-White (extended CIR) models are arbitrage-free models in which the drift parameter $\vartheta(t)$ is a function of time t, estimated so that to match exactly the term structure of the interest rates. The mean-reversion parameter is still assumed to be zero and the volatility parameter is estimated based on historical sample of short-term yields.

The CIR model has the following advantages compared to the Vasicek model.

- Bounded interest rates. Zero interest rates do not have any specific significance in the Vasicek model. The movement of interest rates is conceptually similar for positive and negative interest rates. In the CIR model, the volatility of interest rates decreases to zero as the interest rates approach zero bound. Under certain conditions, the interest rates never cross the zero bound in the CIR model.
- Heteroscedastic volatility of interest rates. The volatility of interest rates in the Vasicek model does not depend on the interest rates level. The interest rate volatility in the high interest rate markets is the same as the volatility in the low interest rate markets. This is typically not consistent with the actual observed markets interest rate behavior.

2.2 Zero-coupon bond prices

The equations for zero-coupon bond prices are used to (i) calibrate the parameters of the interest rate model and to (ii) validate the results of the model numerical estimation. The bullet bond prices, described by the equation (assuming $t = 0$. For arbitrary t, the yield rate r_0 is replaced with r_t and tenor T is replaced with $T - t$).

(2.2) $$P_T = A \times e^{-Br_0}$$

or, equivalently,

(2.3) $$\ln P_T = -B_T \times r_0 + \ln A_T$$

and respective interest rate term structures are summarized in the exhibits below. The formulas are derived in the Appendix A.

Exhibit 2.2 Summary of zero-coupon bond prices

Model name	Zero-coupon bond bullet price	
	$\boldsymbol{B(t,T)}$	$\boldsymbol{A(t,T)}$
Vasicek (1977)	$\frac{1 - e^{-\alpha(T-t)}}{\alpha}$	$e^{-\left[\left(\vartheta - \lambda\sigma - \frac{\sigma^2}{2\alpha}\right) \times \frac{(T-t-B)}{\alpha} + \frac{\sigma^2 B^2}{4\alpha}\right]}$
Hull-White (extended Vasicek, 1990)	$\frac{1 - e^{-\alpha(T-t)}}{\alpha}$	$e^{-\left[\int_t^T [\vartheta(s) - \lambda\sigma] \times B(s) \times ds - \frac{\sigma^2}{2\alpha^2} \times (T-t-B) + \frac{\sigma^2 B^2}{4\alpha}\right]}$
CIR (1985)	$\frac{2 \times \left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \alpha + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma}$	$\left[\frac{2\gamma e^{(\gamma+\alpha+\lambda\sigma)(T-t)/2}}{(\gamma + a + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2\vartheta}{\sigma^2}}$

Model name	Zero-coupon bond bullet price	
	$B(t,T)$	$A(t,T)$
Hull-White (extended CIR, 1990)	$\frac{2\times\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\alpha+\lambda\sigma)\times\left(e^{\gamma(T-t)}-1\right)+2\gamma}$	$e^{-\int_t^T \vartheta(s)\times B(s,T)\times ds}$

where

(2.4)
$$\gamma=\sqrt{(\alpha+\lambda\sigma)^2+2\sigma^2}\to\sigma\sqrt{2+\lambda^2}\text{ as }\alpha\to 0$$

and λ is the market price of risk (with default value assumed to be $\lambda=0$).

Assuming $\alpha=0$, the equations can be simplified as follows

Exhibit 2.3 Summary of zero-coupon bond prices at $\alpha=0$

Model name	Zero-coupon bond bullet price	
	$B(t,T)$	$A(t,T)$
Vasicek (1977)	$T-t$	$e^{-\left[(\vartheta-\lambda\sigma)\times\frac{(T-t)^2}{2}-\frac{\sigma^2}{6}\times(T-t)^3\right]}$
Hull-White (extended Vasicek, 1990)	$T-t$	$e^{-\left[\int_t^T[\vartheta(s)-\lambda\sigma]\times(T-s)ds-\frac{\sigma^2}{6}\times(T-t)^3\right]}$
CIR (1985)	$\frac{2\times\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\lambda\sigma)\times\left(e^{\gamma(T-t)}-1\right)+2\gamma}$	$\left[\frac{2\gamma e^{(\gamma+\lambda\sigma)(T-t)/2}}{(\gamma+\lambda\sigma)\times\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{\frac{2\vartheta}{\sigma^2}}$
Hull-White (extended CIR, 1990)	$\frac{2\times\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\alpha+\lambda\sigma)\times\left(e^{\gamma(T-t)}-1\right)+2\gamma}$	$e^{-\int_t^T \vartheta(s)\times B(s,T)\times ds}$

Respective term structure of the interest rates defined as[6]

(2.5)
$$R_T=-\frac{\ln P_T}{T}$$

is described by the following equation:

(2.6)
$$TR_T=-\ln P_T=-\ln A(t,T)+B(t,T)\times r_0$$

We also use the following notation in this guide: $A(T)=A(0,T)$ and $B(T)=B(0,T)$.

[6] The yield rate of a zero-coupon bond is defined as a value R_T such that $P_T=e^{-TR_T}$.

2.3 Zero-coupon bond price approximations

The equations for zero-coupon bond prices are applied to calibrate parameters of the interest rate option model. The calibration however cannot be described by explicit equations due to complex functional form of the price equations.

To simplify parameter calibration, the mean-reversion parameter is set to zero, $\alpha = 0$. The model with $\alpha = 0$ typically produces robust interest rate option values and is a reasonable assumption in many cases. In practice, the interest rate data supports zero mean reversion in the short-term and positive mean-reversion in the longer-term. Therefore, as an alternative to the $\alpha = 0$ case, we derive in this section the approximate equations for $\alpha \to 0$. In the next section we show how the price approximation can be applied to calibrate the interest rate model parameters.

The approximate equations for zero-coupon bond prices with $\alpha \to 0$ are summarized in the Exhibit 2.4 below.

Exhibit 2.4 Summary of zero-coupon bond price approximations

Model name	Zero-coupon bond bullet price	
	$B(t,T)$	$A(t,T)$
Vasicek (1977) [7,8,9,10]	$(T-t) - \alpha \frac{(T-t)^2}{2}$	$e^{-\left[\left((\vartheta-\lambda\sigma)\frac{(T-t)^2}{2}-\sigma^2\frac{(T-t)^3}{6}\right)-\alpha\times\left((\vartheta-\lambda\sigma)\frac{(T-t)^3}{6}-\sigma^2\frac{(T-t)^4}{8}\right)\right]}$
Hull-White (extended Vasicek, 1990)	$(T-t) - \alpha \frac{(T-t)^2}{2}$	
CIR (1985)		
Hull-White (extended CIR, 1990)		

where

$$\gamma \sim \sigma\sqrt{2+\lambda^2} \tag{2.7}$$

[7] Expression $B(t,T)$ is approximated as $B(t,T) = (T-t) - \alpha\frac{(T-t)^2}{2} + \alpha^2\frac{(T-t)^3}{6} + \alpha^3\frac{(T-t)^4}{24}$

[8] Expression $\frac{(T-t-B)}{\alpha}$ is approximated as follows $\frac{(T-t-B)}{\alpha} \sim \frac{(T-t)^2}{2} - \alpha\frac{(T-t)^3}{6} + \alpha^2\frac{(T-t)^4}{24}$

[9] Expression $\left(\vartheta - \lambda\sigma - \frac{\sigma^2}{2\alpha}\right) \times \frac{(T-t-B)}{\alpha}$ is approximated as follows $\left(\vartheta - \lambda\sigma - \frac{\sigma^2}{2\alpha}\right) \times \frac{(T-t-B)}{\alpha} \sim \left[-\frac{\sigma^2}{4\alpha} \times (T-t)^2\right] + \left[(\vartheta-\lambda\sigma) \times \frac{(T-t)^2}{2} + \sigma^2 \times \frac{(T-t)^3}{12}\right] - \alpha \times \left[(\vartheta-\lambda\sigma) \times \frac{(T-t)^3}{6} + \sigma^2 \times \frac{(T-t)^4}{48}\right]$

[10] Expression $\frac{\sigma^2 B^2}{4\alpha}$ is approximated as follows $\frac{\sigma^2 B^2}{4\alpha} = \left[\frac{\sigma^2}{4\alpha} \times (T-t)^2\right] - \left[\sigma^2 \times \frac{(T-t)^3}{4}\right] + \alpha \times \left[\sigma^2 \times \frac{7}{48} \times (T-t)^4\right]$

2.4 Term structure

The term structure is derived directly from the zero-coupon bond equations using the term structure definition (2.5):

$$R_T = -\frac{\ln P_T}{T} = Br_0 - \ln A$$

The term structure equations for different interest rate models are summarized in the exhibit below.

Exhibit 2.5 Term structure equations

Model name	Mean reversion	Term structure
Vasicek (1977)	$\alpha > 0$	$TR_T = B(0,T) \times r_0 + \int_0^T \vartheta(s) \times B(s,T) \times ds - \frac{\sigma^2}{2\alpha^2} \times \left(T - B(0,T)\right) + \frac{\sigma^2 B(0,T)^2}{4\alpha}$, where $B(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha}$
	$\alpha = 0$	$TR_T = T \times r_0 + \int_0^T \vartheta(s) \times (T-s) \times ds - \frac{\sigma^2}{6} \times T^3 = T \times r_0 + \vartheta \frac{T^2}{2} - -\frac{\sigma^2}{6} \times T^3$,[11] where $B(t,T) \sim (T-t) - \frac{\alpha}{2} \times (T-t)^2 + \frac{\alpha^2}{6} \times (T-t)^3$

2.5 Term structure approximation

[11] Formally, $-\frac{\sigma^2}{2\alpha^2} \times \left(T - B(0,T)\right) + \frac{\sigma^2 B(0,T)^2}{4\alpha} = -\frac{\sigma^2}{2\alpha^2} \times \left(\frac{\alpha}{2} \times T^2 - \frac{\alpha^2}{6} \times T^3\right) + \frac{\sigma^2}{4\alpha} \times \left(T - \frac{\alpha}{2} \times T^2\right)^2 = \frac{\sigma^2}{12} \times T^3 - \frac{\sigma^2}{4} \times T^3 = -\frac{\sigma^2}{6} \times T^3$

Section 3 Parameter Estimation

Parameter estimation is one of the key steps of option estimation. It is arguably a more complex task to produce stable and intuitive estimates for option parameters compared to the option estimation step. The objectives, approaches, and specific equations for option model parameters estimation are presented below.

3.1 Overview

In this section, we overview the criteria which are taken into consideration to assess reasonability of the estimated option model parameters. We present then a high-level overview of the approaches tat can potentially be applied for parameter estimation. Sections 3.3 – 3.7 describe the default parameter estimation methods applied in the option valuation tool. Appendix C provides an overview of alternative parameter estimation approaches.

3.1.1 Objectives

The key objective is to produce parameter values which are intuitive and can be easily interpreted and explained. Specifically, the following considerations are taken into account.

(i) Robustness of estimates to outliers. Daly data is potentially highly sensitive to inaccurate yield data estimates, which can produce highly sensitive parameter estimates.

(ii) Consistency with arbitrage pricing. Interest rate option models are based on arbitrage-free pricing models of interest rates, in which the full interest rate term structure is derived analytically from the short-term interest rate model. Therefore, a reasonably accurate matching of the theoretical and empirical term structures is one of the objectives of the parameter estimation.

(iii) Consistency with market fluctuations. Prepayment option values are expected to be high in the periods of high markets volatility when interest rates increase sharply. It is expected that market interest rates will return back to the equilibrium values and prepayment option will be in the money and exercised.

(iv) Zero interest rate floor considerations. While there is some evidence that interest rates move below zero floor, the evidence is very limited (observed only for government and inter-banking yield rates denominated in European currencies) and interest rate movement below the zero-floor threshold is relatively small. Therefore, zero floor constraint is a reasonable assumption for the interest rates. The implication of the zero-floor assumption is that the call option values are expected to be low for low interest rates (due to limited potential downside exposure of the interest rates).

(v) Mean-reversion modelling. Historically, interest rate models are described reasonably well with a random walk models for the short-term horizon and mean-reversion process for the medium and long-term horizons. Therefore, mean-reversion is a material factor for the interest rate option models.

The consistency of the implemented interest rate option model with the criteria discussed above is discussed in Appendix F.5.

3.1.2 Approaches

The section reviews three alternative approaches to parameter estimation.

- **Regression analysis**. The approach is based on the direct estimation of the interest rate regression model

(3.1) $$dr_t = \mu(t, r_t) \times dt + \sigma(t, r_t) \times dW_t$$

► **Calibration approach**. Under the alternative approach, all parameters of the interest rate model are estimated so that to match market price data described by the equation

(3.2) $$\ln P_T = TR_T = -B(T) \times r_0 + \ln A(T)$$

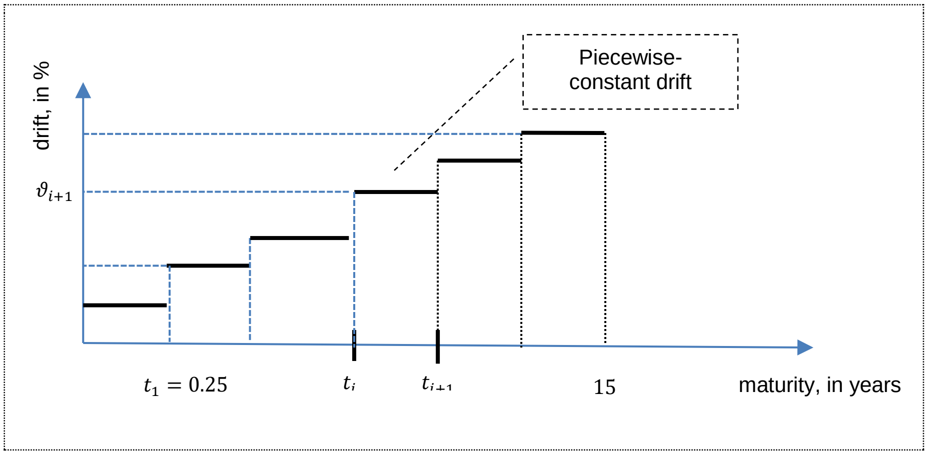
► **Mixed estimation approach**. Under the mixed estimation approach, some parameters are estimated based on historical data and equation (3.1) and other parameters are estimated so that to match current market data and approximate equation (3.2). Specifically, the volatility and mean-reversion parameters are estimated from the equation (3.1) and drift parameter is estimated from the equation (3.2). This is the default approach in this guide.

The data used for parameter calibration is typically represented by bond prices with different maturities, which equivalently is converted to the term structure of the bond yields. In both Hull-White extended Vasicek and CIR model the function ϑ_t can be selected so that to match exactly the term structure of the yield rates. Therefore, the volatility and mean-reversion parameters need to be either calibrated using additional market data or estimated based on historical data. The default parameter estimation approach applied in this guide is to (i) assume constant volatility and mean-reversion parameters and estimate them based on historical data and to (ii) assume either constant or time-varying drift parameter and calibrate the parameter to match the yield term structure.

The interest rate models (such as Hull-White extended Vasicek and extended CIR models), in which the term structure is matched exactly, are referred to as **arbitrage-free** models. Yield term structure is typically estimated based on the Bloomberg or Reuters yield series with different maturity terms. Bloomberg reports the yield series with the following maturity terms (in years): $t_i = \{0.25, 0.5, 1, 2, 3, 4, 5, 7, 8, 9, 10, 15\}$. To match the yield term structure, the drift function is assumed to be piecewise constant estimated for the set of values $\{t_i\}$.

The diagram with the piecewise-constant drift function is illustrated in the diagram below.

Exhibit 3.1 Piecewise-constant drift function



The equation for the yield term structure

$$TR_T = B(T) \times r_0 - \ln A(T)$$

is used to calibrate the parameters of the interest rate process.

We assume in the sections below that the market price of risk is set to zero, $\lambda = 0$ and the mean-reversion parameter is set to zero, $a = 0$. Note that the process with zero mean-reversion does not converge to a long-term equilibrium. Estimation of mean-reversion parameter is discussed in Appendix C.

3.2 Sample parameters

As a first step, a sample is selected which is used to estimate the model parameters. The short-term interest rates are selected based on the short-term yield estimates[12] with the industry sector and credit rating matching the industry sector of the borrowing entity and credit rating of the tested transaction. The sample is selected using the following parameters.

1. Sample size.
2. Period for yield change estimation.
3. Probability threshold for the outlier elimination.

3.3 Volatility parameter

This section describes three alternative methods for volatility estimation adapted for each interest rate model:

1. Constant variance estimation.
    - The advantage of the approach is that (i) it is easy to implement (estimation is based on simple statistics); and (ii) it produces different variance estimates for different credit ratings (typically higher variance for lower ratings capturing higher volatility of yield series with low credit ratings.
    - The potential problems with the approach is that (i) it can potentially be highly sensitive to outliers; and (ii) the same weight is assigned to both recent and remote observations. High market volatility observed a few months ago will still have a large impact on the volatility estimate even though current market volatility may be low. The volatility estimates are often non-stable.
2. Variance based on market volatility index.
    - The advantage of the approach is that (i) it produces stable and reasonable estimates; (ii) high / low variance estimates match high / low market volatility; (iii) implied volatility estimated based on swaption market price data is a preferred approach from the transfer pricing perspective; and (iv) volatility index show consistent behavior over time and is estimated by a reputable source.
    - The key problem with the approach is that it assigns the same volatility to each credit rating. In practice, the interest rate volatility index is likely implied from the option prices on US$ Treasuries. The volatility estimate also assumes Normal model of interest rates (Vasicek model). Therefore, it is not applicable to CIR or other types model types.
3. Variance based on EWMA and GARCH(1, 1) models.

[12] Typically, 3-month yield series is selected for the estimation of the option model parameters.

- ► The advantage of the approach is that (i) it assigns higher weight to recent volatility (and, therefore, variance estimates move cyclically with the market volatility); and (ii) the volatility estimate depends on the yield series credit rating.
- ► Potential problems with the approach is that (i) it is more difficult to estimate (it's based on 1st – order auto-regression model); and (ii) the results can potentially be sensitive to outliers (**check**).

The default approach is to estimate sample volatility using annual deviations in the yield rates. Alternative approaches are discussed in Appendix C.1. The default approach was selected for the following reasons:

(vi) The estimator is robust with respect to the outliers.

(vii) The estimator captures period of high volatility and high interest rates through higher volatility parameter (check EWMA again as a default option).

3.3.1 Constant variance

Under the 'constant variance' approach, the variance is estimated as the following sample statistics.

$$\sigma_n = \sigma = \kappa \times stdev\,[u_t] \tag{3.3}$$

where κ is the normalization parameter applied to produce an annual standard deviation parameter. Specifically, if u_t is estimated using annual changes in the data, then $\kappa = 1$. If u_t is estimated using daily changes in the data, then $\kappa = \sqrt{250}$ (assuming there are 250 business days in the year). In general, parameter κ is estimated using the following equation:

$$\kappa = \sqrt{\frac{250}{\tau}} \tag{3.4}$$

where τ is the number of business days between consecutive yield data observations applied to estimate the changes in the residuals.

Parameter τ is selected based on the following considerations: (i) presence of outliers in the sample; and (ii) period over which volatility is estimated. The larger is parameter τ, the lower is the impact of outliers on the results and the longer is the period over which the volatility is estimated.

3.3.2 Vasicek and Hull-White (extended Vasicek)

Sample volatility is estimated based on the following short rate representation.

$$dr_t = (\vartheta_t - ar_t) \times dt + \sigma \times dW_t$$

The annual volatility σ is estimated based on annual deviations of the yield rates by applying the following equation (assuming zero mean-reversion, $a = 0$, and constant drift, $\vartheta_t = \theta$):

$$\varepsilon_t = \sigma \sum\nolimits_{t-250}^{t} dW_t \times \sqrt{dt} = \sum\nolimits_{t-250}^{t} (dr_t - \theta dt) = r_t - r_{t-250} - \theta \tag{3.5}$$

where 250 is the proxy for the number of business days in a year. The volatility parameter σ is estimated as a sample standard deviation of the sample constructed using equation (3.5).[13]

3.3.3 CIR and Hull-White (extended CIR)

Sample volatility is estimated based on the following short rate representation.

$$dr_t = (\vartheta_t - ar_t) \times dt + \sigma \times \sqrt{r_t} \times dW_t$$

The annual volatility σ is estimated based on annual deviations of the yield rates by applying the following equation (assuming zero mean-reversion, $a = 0$, and constant drift, $\vartheta_t = \theta$):

$$\varepsilon_t = \sigma \sum\nolimits_{t-250}^{t} dW_t \times \sqrt{dt} = \sum\nolimits_{t-250}^{t} \frac{dr_t - \theta dt}{\sqrt{r_t}} \tag{3.6}$$

The volatility parameter σ is estimated as a sample standard deviation of the sample constructed using equation (3.6). For simplicity, parameter θ is set to zero in the above equation.

3.4 Drift parameter

Drift parameter is estimated to match approximately (in Vasicek and CIR) or exactly (in Hull-White extended Vasicek or Hull-White extended CIR) the term structure of the yield rates. The estimation approach was selected as the default approach to produce consistency between the option valuation and the term premium. Importance of consistency for put options from the arbitrage pricing perspective is illustrated in Appendix G.4.1.2. A similar argument can be applied for the call options. An increasing term structure implies that in the initial periods of the option life the interest rates are below the coupon rate (which includes the term premium component). Therefore, the increasing term structure can be viewed as an effective barrier to exercise the call option early and the barrier reduces the value of the call option. The steeper is the barrier, the larger is the negative impact on the call option value.[14]

The equations for the Vasicek and CIR model are presented below. The equations for Hull-White (extended Vasicek) and Hull-White (extended CIR) models are presented in Appendix C.2.

3.4.1 Vasicek model

The interest rate term structure is described by the following equation

$$TR_T = B(T) \times r_0 + \int_0^T \vartheta(s) \times B(s,T) \times ds - \frac{\sigma^2}{2\alpha^2} \times \big(T - B(T)\big) + \frac{\sigma^2 B(T)^2}{4\alpha}$$

where $B(T) = \frac{1-e^{-\alpha T}}{\alpha}$. The equation can also be represented as follows.

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[13] Parameter θ does not have an impact on the standard deviation and for simplicity is set to zero.

[14] As illustrated in Appendix G.4.1.2, the impact on the put option is the opposite. The larger is the term structure, the higher is the incentive to exercise the put option early and get the benefit of the term premium. Therefore, the higher is the term premium, the larger is the positive impact on the put option value.

$$\int_0^T \vartheta(s) \times B(s,T) \times ds = TR_T - r_0 \times B(T) + \frac{\sigma^2}{2\alpha^2} \times \left(T - B(T)\right) - \frac{\sigma^2 B(T)^2}{4\alpha} = G(T)$$

where

(3.7) $$G(T) = [TR_T - B(T) \times r_0] + \frac{\sigma^2}{2\alpha^2} \times \left(T - B(T)\right) - \frac{\sigma^2 B(T)^2}{4\alpha}$$

As $\alpha \rightarrow 0$, the function $G(T)$ converges to the following function[15]

(3.8) $$G(T, \alpha \rightarrow 0) = T(R_T - r_0) + \frac{\sigma^2 T^3}{6}$$

If the drift parameter ϑ is constant, then equations (3.8) and (C.10) are estimated for the maturity term $t_{i+1} = T$ only ($t_i = 0$) and the drift parameter is described by the following equations.

(3.9) $$\vartheta = \alpha \frac{G(T)}{T - B(T)}$$

for $\alpha > 0$ and

(3.10) $$\vartheta = 2\frac{G(T)}{T^2} = 2\frac{(R_T - r_0)}{T} + \frac{\sigma^2 T}{3}$$

for $\alpha = 0$.

Equation (3.10) can also be validated by estimating parameter ϑ directly from the Vasicek zero-coupon bond price equation described in the Exhibit 2.2.[16]

3.4.2 CIR model

The interest rate term structure is described by the following equation

$$TR_T = B(T) \times r_0 + \int_0^T \vartheta(s) \times B(s) \times ds$$

where $B(T) = \frac{2\times(e^{\gamma T}-1)}{(\gamma+\alpha)\times(e^{\gamma T}-1)+2\gamma}$. Similar to the Vasicek model, the equation can be represented as

$$\int_0^T \vartheta(s) \times B(s) \times ds = G(T)$$

---

[15] $\frac{\sigma^2}{2\alpha^2} \times \left(T - B(T)\right) - \frac{\sigma^2 B(T)^2}{4\alpha} \rightarrow \frac{\sigma^2}{2\alpha^2} \times \left(\alpha\frac{T^2}{2} - \alpha^2\frac{T^3}{6}\right) - \frac{\sigma^2}{4\alpha} \times (T^2 - \alpha T^3) = -\frac{\sigma^2 T^3}{12} + \frac{\sigma^2 T^3}{4} = \frac{\sigma^2 T^3}{6}$

[16] $TR_T = -\ln A(T) + B(T) \times r_0 = \left(\vartheta - \frac{\sigma^2}{2\alpha}\right) \times \frac{(T-B)}{\alpha} + \frac{\sigma^2 B^2}{4\alpha} + B(T) \times r_0$ or $\vartheta = \alpha \frac{[TR_T - B(T)\times r_0] + \frac{\sigma^2}{2\alpha^2}(T-B) - \frac{\sigma^2 B^2}{4\alpha}}{T-B} = \alpha\frac{G(T)}{T-B}$

where

$$G(T) = TR_T - B(T) \times r_0 \tag{3.11}$$

As before, suppose that $t_0 = 0, t_1, \ldots, t_{n-1}, t_n = T$ are discrete periods with observed market yield rates R_i and respective values of the function $G(T)$ equal to G_i.

In the case of constant parameter $\vartheta(t) = \vartheta$, parameters t_i and t_{i+1} are set to $t_i = 0$ and $t_{i+1} + T$. The equations (3.11) and (C.11) can be represented then as follows

$$\theta = \frac{\gamma^2 - \alpha^2}{2} \times \frac{TR_T - B(T) \times r_0}{-(\gamma + \alpha) \times T + 2 \times (\ln[(\gamma + \alpha) \times e^{\gamma \mathrm{T}} + (\gamma - \alpha)] - \ln 2\gamma)} \tag{3.12}$$

and

$$\theta = \frac{\gamma^2}{2} \times \frac{TR_T - B(T) \times r_0}{-\gamma T + 2 \times (\ln[e^{\gamma T} + 1] - \ln 2)} \tag{3.13}$$

Equation (3.13) can also be validated by estimating parameter ϑ directly from the CIR zero-coupon bond price equation described in the Exhibit 2.2.[17]

3.5 Mean-reversion parameter

There are currently two default approaches to set the mean-reversion parameter

(i) Calibrate the mean-reversion parameter to match a long-term equilibrium yield rate.

(ii) Estimate mean-reversion parameter based on Hodrick-Prescott (**HP**) filter model.

(iii) Set mean-reversion parameter to zero, which corresponds to the random walk model of the yield rates. Note that random walk model is non-stationary and does not have a long-term equilibrium.

The two approaches are discussed below. Oher approaches are discussed in Appendix C.3.

3.5.1 Mean-reversion parameter calibrated to long-term equilibrium

Under the approach, mean-reversion parameter is estimated from the following equation, which relates mean-reversion parameter to the interest rate long-term equilibrium value.

$$r^* = \frac{\theta}{\alpha} \tag{3.14}$$

Note that under the approach, the estimation of the mean-reversion parameter is effectively reduced to estimation of the interest rate long-term equilibrium value. The long-term equilibrium can be estimated for example using x-year moving average of the interest rates.

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[17] $TR_T = -\ln A(T) + B(T) \times r_0 = -\frac{2\vartheta}{\sigma^2} \times \ln \frac{2\gamma e^{(\gamma+\alpha)T/2}}{(\gamma+a)\times(e^{\gamma T}-1)+2\gamma} + B(T) \times r_0$ or $\vartheta = \frac{\sigma^2}{2} \times \frac{TR_T - B(T)\times r_0}{\ln[(\gamma+a)\times(e^{\gamma T}-1)+2\gamma] - \ln 2\gamma - \frac{(\gamma+\alpha)T}{2}} = \frac{\gamma^2-\alpha^2}{2} \times \frac{TR_T - B(T)\times r_0}{-(\gamma+\alpha)T + 2\times[\ln(\gamma+a)e^{\gamma T} + (\gamma-a) - \ln 2\gamma]}$

3.5.2 Mean-reversion parameter based on Hodrick-Prescott filter model

The mean reversion parameter is estimated based on the following equation

$$dr_t = a \times (\tilde{r}_t - r_t) \times dt + \sigma(r_t) \times dW_t \tag{3.15}$$

where $\tilde{r}_t$ is interpreted as the long-term trend component and a is the mean reversion parameter.[18] The equation is using HP filter as follows.

(i) Estimate long-term and cyclical component of the yield series, denoted respectively as $\tilde{r}_t$ and $\varepsilon_t^{cyc} = \tilde{r}_t - r_t$.

(ii) Estimate linear regression $dr_t = \alpha \times \varepsilon_t^{cyc} \times dt + \sigma(r_t) \times dW_t$. The estimate of the α parameter in the linear regression equation is the model mean-reversion parameter.

(iii) Estimate volatility parameter $\sigma(r_t)$ based on the residuals of the linear regression model.

The time-varying long-term trend $\tilde{r}_t$ in the interest rate model is replaced with a constant equilibrium value r^*. There are three alternative approaches to select the r^* parameter:

(i) Set r^* equal to the most recent long-term trend parameter $\tilde{r}_t$

(ii) Set $r^* = \frac{\theta}{\alpha}$, where drift parameter θ is estimated as discussed in the previous section.

(iii) Set r^* based on expert judgement (for example, set it equal to the long-term average interest rate)

Note that estimation of the long-term equilibrium is effectively equivalent to the estimation of drift parameter, which are related through the equation (3.14).[19]

3.5.3 Random walk model

Under the random walk model of the interest rates, the mean-reversion parameter is set to zero.

3.6 Coupon rate

Original coupon rate c is estimated based on the observed term structure. However, after calibrating the volatility and drift parameters and generating the corresponding theoretical term structure described by equation (hw.vsk.1), the bullet price of the bond may not match the bond par value. The coupon rate c^* is solved for implicitly so that the bullet value of the bond equals to the bond par value.[20,21]

The coupon rate is estimated based on the following equation of the bond par value:

[18] Mean –reversion parameter must be positive so that the interest rate reverts to the long-term trend whenever it is above/below the trend.

[19] Equation (3.14) can be viewed not only as the equation that relates mean-reversion to the long-term equilibrium, but also as an equation which relates drift to the long-term equilibrium.

[20] Note that if the bond bullet value is different from the par value, it may be optimal to exercise the bond in period $t = 0$ immediately at the bond issue date. The adjustment of the coupon rate is performed to rule out the above cases.

[21] An alternative approach is to shift the term structure of the interest rates. However, in the case of CIR and Hull-White (extended CIR) models a shift in the term structure results in a different volatility function and as a result the distribution over the interest rate tree must also be recalculated. Calibration of the coupon rate is a more efficient approach to ensure that the bond is priced at par at $t = 0$.

$$\sum_t D_t \times C_t + \sum_t D_t \times P_t = 100$$

where C_t are bond coupon payments, P_t are bond principal repayments, and D_t are zero coupon prices. The coupon estimation approach depends on the structure of the bond cash flows. Two cases are considered:

- **Case A**: The bond has interest deferral provision and accrued interest is capitalized over the interest deferral period. In this case, the cash flow is a non-linear function of the bond coupon rate and therefore cannot be solved explicitly. The coupon rate that generates the bond par value is estimated numerically by solving the above implicit equation for the coupon rate. At each iteration the expression $\sum_t D_t \times C_t + \sum_t D_t \times P_t$ is estimated by running a backward recursion procedure.
- **Case B**: the bond cash flow structure is different from the structure described in Case A. In this case, the cash flow is a linear function of the bond coupon rate. Therefore, the coupon rate that generates the bond par value can be estimated as follows:

$$c^* = c_0 \times \left[\frac{100 - \sum_t D_t \times P_t}{\sum_t D_t \times C_{0,t}} \right]$$

  where $c_0 > 0$ is an arbitrary coupon rate. The equation is estimated by (i) running backward recursion for the repaid principal cash flows P_t to estimate $\sum_t D_t \times P_t$ numerator; (ii) running backward recursion for the coupon cash flow payment $C_{0,t}$ to estimate $\sum_t D_t \times C_{0,t}$ denominator; and (iii) applying the above equation.

In the case of Vasicek or Hull-White (extended Vasicek) models, the term structure adjustment can be used as an alternative to the coupon adjustment. The option is selected whenever the coupon must be set fixed at the actual bond coupon payment.

3.7 Summary

Parameter estimation procedure is summarized as follows.

- **Model selection**. The formulas applied to estimate the parameters of the model depend on the selected model.
- **Variance**. The variance parameter is estimated based on the historical sample of short-term yield rates
  - using equation (3.5) for the Vasicek and Hull-White (extended Vasicek) models; and
  - using equation (3.6) for the CIR and Hull-White (extended CIR) models.
- **Drift**. The drift parameter is estimated to approximate the term structure linear slope using the following equations.
  - Equations (C.10) for the Hull-White (extended Vasicek) model;
  - Equations (3.9) – (3.10) for the Vasicek model;
  - Equations (C.11) for the Hull-White (extended CIR) model; and
  - Equations (3.12) – (3.13) for the CIR model.
- **Mean-reversion**. The mean reversion parameter is either
  - calibrated to match the interest rate long-term equilibrium;
  - estimated using HP filter model; or

- ► is set to zero assuming a random walk model of interest rates.

► **Coupon rate**. The coupon rate is estimated numerically so that the price of the bullet bond is equal to par value.

Conditional on the estimated parameters, zero-coupon bond prices and the term structure of the interest rate process are estimated as follows.

► **Zero-coupon bond prices**. Prices of zero-coupon bond prices are estimated as

$$P(T) = A(T) \times e^{-B(T)r_0}$$

where functions $A(T)$ and $B(T)$ are described for different processes in the Exhibit 2.2. More detailed equations for the function $A(T)$ for the case of the Hull-White models are provided in the Appendix A. Specifically

- ► Equations (A.5) – (A.7) for $\alpha > 0$ and equations (A.8) – (A.10) for $\alpha = 0$ parameter of the Hull-White (extended Vasicek) model; and
- ► Equations (A.14) – (A.16) for $\alpha > 0$ and equations (A.17) – (A.18) for $\alpha = 0$ parameter of the Hull-White (extended CIR) model;

► **Term structure**. The term structure is derived directly from the zero-coupon bond prices using the equation below

$$R(T) = -\frac{\ln P(T)}{T}$$

Section 4 Numerical Modelling

The steps in the numerical modelling of the interest rate options can be summarized as follows.

1. Estimate the discount factors applied in the interest rate option numeric calculations;
2. Estimate the tree of interest rate states and probability distribution over the states, which is derived from the respective interest rate stochastic process;
3. Estimate the bond cash flows;
4. Estimate numerically the bond value (both the bullet value and the callable/putable bond value).

Each step of the modelling process is summarized below.

4.1 Discount factors

Discount factors are calculated as

$$D[dt] = A \times e^{-Br_0}$$

For a selected (small) time increment dt the discount factor can be approximated as

$$D[dt] = e^{-dt \times r_0}$$

In practice selected tree step dt may not be small (the default value in the option calculation tool is $dt = 0.25$). Therefore, it is preferable to use actual bullet prices as discount factors, which depend on the selected family of the interest rates. The equations for the Vasicek and CIR bullet prices were summarized in Sections 2.2 and 2.3.

4.2 Interest rate tree estimation

The interest rate tree estimation includes two steps: (i) construction of the tree states and estimation of the tree states probabilities. The steps are summarized in more detail below.

1. Construction of the interest rate tree states
   - Discretize the time/space set of interest rate states.
   - To discretize the time a tree step is selected (by default the tree step is set to three-months period, $dt = 0.25$). The grid of discrete time periods is set to be uniform.
   - The minimum and maximum bounds of the interest rates are estimated. The bounds are estimated in such a way that the probability of the interest rate process to move outside the bounds is smaller than some small threshold value.
   - The grid of interest rates is constructed within the estimated bounds. For the Vasicek and Hull-White (extended Vasicek) models the grid is set uniform. For the CIR and Hull-White (extended CIR) models the grid is constructed so that $\frac{\Delta r_t}{\sqrt{r_t}}$ are distributed uniformly.
2. Estimation of the tree states probabilities.
   - The set of states in period $t = 0$ consists of a single state r_0, which is assigned probability one.

- For each period t and state $r_{t,i}$, transition probabilities $q_j = Q_{i,j}(t, t+dt)$ are constructed using the mean and standard deviation functions $\mu(t, r_t)$ and $\sigma(t, r_t)$ of the selected interest rate process.
- Due to the fact that the states $r_{t+dt,j}$ are discrete, the numerical mean and standard deviation of the change in interest rates $dr_{t,i}$ will generally be different from the values of $\mu(t, r_t)$ and $\sigma(t, r_t)$ functions. We apply two alternative approaches to adjust the transition probabilities.

  Under the default "contraction mapping" approach, parameters $\mu(t, r_t)$ and $\sigma(t, r_t)$ are adjusted to parameters $\tilde{\mu}(t, r_t)$ and $\tilde{\sigma}(t, r_t)$ such that the mean and standard deviation parameters of the discrete distribution matches the actual parameters $\mu(t, r_t)$ and $\sigma(t, r_t)$.

  Under an alternative approach, the transition probabilities q_j estimated using actual parameters $\mu(t, r_t)$ and $\sigma(t, r_t)$ are adjusted to new probabilities $\tilde{q}_j$ such that under the adjusted probabilities (i) the mean and standard deviation parameters of the discrete distribution matches the actual parameters $\mu(t, r_t)$ and $\sigma(t, r_t)$ and (ii) the distance between probabilities $\tilde{q}_j$ and q_j is minimized.

  Because the transition probabilities are adjusted to match the theoretical transition probabilities functions, the estimated numerical distribution of interest rates must match closely the theoretical distributions. Therefore, the validation of the numerical calculations using the interest rate distribution parameters (described in Section 5) should produce very close numbers between the numerical and theoretical mean and standard deviation estimates.

  A more detailed discussion of modelling transition probabilities is provided in the Appendix B.3.

3. Remove the states with the estimated transition probabilities to reach the state below a certain small threshold value.

4.3 Estimation of bond cash flows

The next step is to estimate the cash flows in the bond transaction paid by the borrower (received by the lender) assuming two scenarios: (i) the bond is outstanding and (ii) the bond is redeemed.

If the bond is not redeemed, then the cash flows may include the following components (which can be set in the option tool).

- Bond coupon payments. The coupon rate and coupon payment frequency must be set to estimate the coupon payments;
- Interest deferral provision. If the bond has interest deferral provision, then the interest payments are assumed to be capitalized and repaid after the termination of the interest deferral period;
- Bond principal repayment.
- Bond amortization schedule.

If the bond is redeemed, then the redemption value is estimated based on the specified redemption terms.

- If the bond is redeemed prior to the make-whole provision termination date, then the cost of calling the bond is assumed to be infinite in the option tool. The option will never be exercised prior to the make-whole termination date;
- If the bond is redeemed after the make-whole provision termination date, then the redemption value is estimated based on the provided redemption penalty structure. If there is no penalty, then the redemption value equals the sum of bond principal value and the accrued interest amount.

As discussed in the previous section, bond coupon rate is an endogenous parameter in the option tool and is estimated so that the bullet value of the generated bond cash flows equals to the bond par value.

4.4 Option price estimation

The steps of the option estimation procedure are summarized as follows:

1. Estimate forward prices:

$$P_{t,i}^{\tau,*} = D_{t,i} \times \sum_j Q_{i,j}(t, t+dt) \times P_{t+dt,i}^{\tau-dt,*} \tag{4.1}$$

   where $P_{t,i}^{0,*} = 1$. The forward prices are applied for the following reasons:

   - Estimate bullet prices
   - Estimate the term structure
   - Estimate the option value after the option is exercised for a non-zero notice period.

2. Estimate the bond bullet value. The bond bullet value is calculated using the backward recursion. The bond value is estimated first at the maturity date $t = T$. The value is then estimated backwards. Assuming that the value is estimated in period $t + dt$, the value in period t is estimated using the following equation

$$P_{t,i}^{*} = c_t \times dt + D_{t,i} \times \sum_j Q_{i,j}(t, t+dt) \times P_{t+dt,i}^{*} \tag{4.2}$$

   where $P_{t,i}^{*}$ is the bond bullet price, $D_{t,i}$ is the discount factor, c_t is the cash flow paid by the bond, and $Q_{i,j}(t, t+dt)$ are interest rate transitional probabilities. Alternatively, the bullet prices can be estimated using the forward prices:

$$P_{t,i}^{*} = \sum_{\tau=1}^{T} P_{t,i}^{\tau,*} \times c_\tau \times dt \tag{4.3}$$

   The bullet value will generally be different from the bond par value. As a result, it is potentially possible that it is optimal to exercise the option in period $t = 0$.

3. Shift the term structure of interest rates (by modifying parameter r_0 in the term structure equation) so that the bond bullet value equals the par value. Note that the discount rates $D_{t,i}$ (and respectively the bond bullet price) decrease uniformly with the increase in r_0. Therefore, there is a unique value of r_0 such that the bond bullet value equals the par value.

4. Estimate the callable (putable) bond value. The borrower minimizes the cost of the bond by solving the following optimization problem in each period t and state r_i.

$$P_{t,i} = \min\left[\hat{P}_{t,i}, c_t \times dt + D_{t,i} \times \sum_j Q_{i,j}(t, t+dt) \times P_{t,i}\right] \tag{4.4}$$

   where $P_{t,i}$ is the callable bond price, and $\hat{P}_{t,i}$ is the bond redemption value.

In the case of the putable bond, the lender maximizes the bond value. Therefore, the above equation is modified accordingly. Note that as part of the option value calculation, the algorithm also estimates the set of states in which the option is exercised / not exercised. Formally we define the function indicator of the interest rate tree states in which the option is exercised (not exercised) as follows:

(4.5)
$$\mathcal{A}(t, r_i) = \begin{cases} 1, if\ the\ bond\ is\ redemed\ instate\ (t, r_i) \\ 0, otherwise \end{cases}$$

5. Estimate the value of the option as the difference between the value of the callable (putable) bond and the value of the bullet bond (bond par value).

The equation (4.4) is solved using standard backward recursion methods, starting from period $t = T$ (when the bond principal is fully repaid and the price $P_{T,i}$ equals the bond redemption value, and moving backwards to period $t = 0$. The backward recursion in the ac.finance.SRM tool is designed to maximize the objective function. Therefore, for the callable bonds the cash flows and the bond redemption functions are reversed to the negative sign and minimization is reversed to maximization. (No change is required for the putable bonds).

4.5 Option premium (discount) estimation

The option price estimated above is a fixed price that should be accounted for as a bond price discount for callable bond (premium for putable bond). However, in the interest benchmarking analysis the price should be converted to the bond interest rate premium for callable bond (interest rate discount for putable bond).

Note that the payment of the interest rate premium (discount) is conditional on the fact whether the bond was redeemed or not. Suppose that A^0 is the price of the security that pays dt cash flow in each state of the process in which the option is not exercised (each state (t, r_i) such that $\mathcal{A}(t, r_i) = 0$. The value of A^0 is referred to as the **annuity adjustment factor**. The value of A^0 can be calculated numerically using the backward recursion procedure described by the following equation:

$$A^0_{t,i} = \begin{cases} 0, & if\ t = 0\ or\ option\ was\ exercised\ prior\ to\ t \\ dt, & otherwise \end{cases} + \sum_{j:\mathcal{A}(t+dt, r_j)=0} D_{t,j} \times Q_{i,j}(t, t+dt) \times A^0_{t+dt,j}$$

where $\mathcal{A}^0_{t,i} = \begin{cases} 0, & if\ t = 0\ or\ option\ was\ exercised\ prior\ to\ t \\ dt, & otherwise \end{cases}$ is the objective function used in the backward recursion procedure. The option premium (discount) π is calculated as follows.

(4.6)
$$\pi = \frac{P}{A^0}$$

where P is the option price and A^0 is the annuity adjustment factor.

Section 5 Model Validation

Numerical estimation of the call or put option involves the following steps:

1. Construction of the interest rate tree, which models stochastic movement in the interest rates;
2. Estimation of the bond bullet value (assuming no option is present);
3. Estimation of the bond value in the presence of the option assuming that the borrower (lender) chooses whether to exercise the option so that to minimize the cost of the bond (maximize the bond value);

Whenever possible the numerical calculations should be validated against theoretical values. In the case of the option estimation algorithm, the following components of the numerical calculations can be validated:

1. **Interest rate distribution**. Construction of the interest rate tree involves two steps: (i) construction of the discrete set of tree states, and (ii) estimation of the interest rate distribution over the discrete set of states. After the interest rate tree is constructed, the mean and standard deviation of the tree states are estimated for each period based on the constructed discrete set of states and state probabilities. The numerical distribution of states can be validated against the theoretical distributions derived in section 4.1. In practice we validate the state distribution only at the maturity term $t = T$.
2. **Implied model parameters**. In practice it may difficult to interpret how material is the deviation of the terminal distribution mean and standard deviation from the theoretical values. As part of the model validation process, the terminal distribution mean and standard deviation are converted into the implied theoretical model parameters. The implied model parameters are compared then with the actual values used in the model estimation. The details of implied parameters estimation are provided in the Appendix B.3 and are summarized in the exhibit of Section 5.2..
3. **Bond bullet prices**. Bond bullet prices are estimated numerically as part of the option calculations. The formulas for the bond bullet prices were also provided in Section 2. The numerical estimates of the bond bullet prices can be validated against their theoretical values.
4. **Zero volatility parameter**.
5. **DerivaGem tool**. DerivaGem is an option tool developed by John Hull and described in detail in the Appendix E. DerivaGem implements Hull-White(extended Vasicek) model and therefore the output of this interest rate option valuation tool can be validated directly against the output of the DerivaGem tool. The DerivaGem Lognormal model does not correspond to Hull-White(extended CIR) model and therefore cannot be used to validate it.

5.1 Terminal distribution parameters

To validate the numerically estimated distributions of the interest rates against the theoretical values, we need to derive the average and standard deviation of the interest rate distribution for each period t (including $t = T$). The formulas for the interest rate distribution for different families of interest rate models are summarized in the exhibit below.

Exhibit 5.1 Summary of interest rate distribution parameters

Model name	Mean	Variance
Vasicek (1977)	$\mu = \frac{\vartheta}{\alpha} + e^{-\alpha t} \times \left(r - \frac{\vartheta}{\alpha}\right)$ in the case $\alpha = 0$ $\mu = r + \vartheta t$	$\nu_t = \frac{\sigma^2}{2\alpha} \times (1 - e^{-2\alpha t})$ in the case $\alpha = 0$ $\nu_t = \sigma^2 \times t$
Hull-White (extended Vasicek, 1990)	$\mu = e^{-\alpha t} \times \left[r + \int_0^t e^{\alpha s}\, \vartheta(s) ds\right]$ in the case $\alpha = 0$ $\mu = r + \Theta(t)$	$\nu_t = \frac{\sigma^2}{2\alpha} \times (1 - e^{-2\alpha t})$ in the case $\alpha = 0$ $\nu_t = \sigma^2 \times t$
Cox – Ingersoll – Ross (1985)	$\mu = \frac{\vartheta}{\alpha} + e^{-\alpha t} \times \left(r - \frac{\vartheta}{\alpha}\right)$ in the case $\alpha = 0$ $\mu = r + \vartheta t$	$\nu_t = \sigma^2 \times \left[\frac{r}{\alpha} \times e^{-\alpha t} \times (1 - e^{-\alpha t}) + \frac{\vartheta}{\alpha^2} \times \left(\frac{1}{2} - e^{-\alpha t} + \frac{1}{2} e^{-2\alpha t}\right)\right]$ in the case $\alpha = 0$ $\nu_t = \sigma^2 \times \left[rt + \vartheta \times \frac{t^2}{2}\right]$
Hull-White (extended CIR, 1990)	$\mu = e^{-\alpha t} \times \left[r + \int_0^t e^{\alpha s}\, \vartheta(s) ds\right]$ in the case $\alpha = 0$ $\mu = r + \int_0^t \vartheta(s) ds$	$\nu_t = \frac{\sigma^2 \times e^{-2\alpha t}}{\alpha} \times \left[r \times (e^{\alpha t} - 1) + \int_0^t e^{au} \times (e^{at} - e^{au}) \times \vartheta_u\, du\right]$ in the case $\alpha = 0$ $\nu_t = \sigma^2 \times \left[rt + \int_0^t (t - u)\vartheta_u du\right]$

As discussed above, the transition probabilities in the numerical model are adjusted so that to approximate closely theoretical $\mu(t, r_t)$ and $\sigma(t, r_t)$ parameters of the selected interest rate process. Therefore, the parameters of the numerically calculated interest rate distribution should match closely the theoretical parameters summarized in the Exhibit 5.1 above.

For the stepwise constant drift function, the above equations with the integrals replaced by respective summations are provided in the Appendix B. Appendix B also provides details on how to derive the above equations.

5.2 Implied model parameters

The implied parameters are estimated as follows.

- ► First, we estimate the implied drift parameter based on the terminal distribution mean value. Note that since Hull-White (extended Vasicek and CIR) models have a time-dependent drift parameter, we estimate the average value of the drift parameter. The estimates for the constant implied drift parameter are described in the Appendix B.1.3 by equation (B.3) for $\alpha > 0$ and equation (B.4) for $\alpha = 0$ and are summarized in the Exhibit 5.2 below.
- ► Next, we estimate the implied volatility parameter conditional on the estimated drift parameter. In the case of Hull-White (extended Vasicek and CIR) models, we substitute the average constant

drift parameter in the terminal distribution variance equation and derive the respective implied variance parameter.

- In the case of Vasicek and Hull-White (extended Vasicek) models the implied volatility parameters are described by equation (B.5) for $\alpha > 0$ and equation (B.6) for $\alpha = 0$ and are summarized in the Exhibit 5.2 below.
- In the case of CIR and Hull-White (extended CIR) models the implied volatility parameters are described by equation (B.9) for $\alpha > 0$ and equation (B.10) for $\alpha = 0$ and are summarized in the Exhibit 5.2 below

The equations for the implied model parameters are summarized in the Exhibit 5.2 below.

Exhibit 5.2 Summary of interest rate distribution parameters

Model name	Mean	Variance
Vasicek (1977) and Hull-White (extended Vasicek, 1990)	$\vartheta = \alpha \times \frac{\mu_T - re^{-\alpha T}}{1 - e^{-\alpha T}}$ in the case $\alpha = 0$ $\vartheta = \frac{\mu_T - r}{T}$	$\sigma = \sigma_T \times \sqrt{\frac{2\alpha}{1 - e^{-2\alpha T}}}$ in the case $\alpha = 0$ $\sigma = \sigma_T \times \sqrt{\frac{1}{T}}$
Cox – Ingersoll – Ross (1985) and Hull-White (extended CIR, 1990)	$\vartheta = \alpha \times \frac{\mu_T - re^{-\alpha T}}{1 - e^{-\alpha T}}$ in the case $\alpha = 0$ $\vartheta = \frac{\mu_T - r}{T}$	$\sigma = \frac{\sigma_T}{\sqrt{\frac{r}{\alpha} \times e^{-\alpha t} \times (1 - e^{-\alpha t}) + \frac{\vartheta}{\alpha^2} \times \left(\frac{1}{2} - e^{-\alpha t} + \frac{1}{2} e^{-2\alpha t}\right)}}$ in the case $\alpha = 0$ $\sigma = \frac{\sigma_T}{\sqrt{rT + \vartheta \times \frac{T^2}{2}}}$

5.3 Validation against deterministic model

The objective of the validation approach is to simplify the model so that the stochastic model is converted to a deterministic model which can be estimated simple NPV calculations. Basically, the objective of the validation approach is to ensure consistency of bond valuation with the limit deterministic approach. Note that since the deterministic approach is in many cases a default approach to valuation, it is important to ensure consistency in valuation and understand the source of valuation discrepancies (if any).

To convert the stochastic into a deterministic model, we consider a limit case $\sigma = 0$. For simplicity we also assume $\alpha = 0$. Under the parameters both Hull-White extended Vasicek and CIR interest rate models are described by the following equation:

(5.1) $$dr_t = \vartheta_t \times dt$$

or equivalently

(5.2) $$r_t = \int_0^t \vartheta_s \times ds$$

The equation (5.2) models forward market rates. Zero-coupon prices are modelled as

(5.3) $$P_t = e^{-\int_0^t r_s \times ds} = e^{-\int_0^t \left[\int_0^u \vartheta_u \times du\right] \times ds} = e^{-\int_0^t \vartheta_s \times (t-s) \times ds}$$

Yield rates are described respectively by the following equation

(5.4) $$R_t = -\frac{\ln P_t}{t} = \frac{\int_0^t \vartheta_s \times (t-s) \times ds}{t}$$

Validation algorithm.

(i) Drift parameter

- ► Estimate the term structure $\{y_j\}$ used in the analysis
- ► Estimate deterministic discount rates using the equation below

$$P_j = \frac{1 - y_j\left[\Delta_1 P_1 + \cdots + \Delta_{j-1} P_{j-1}\right]}{1 + y_j \Delta_j}$$

and

$$P_1 = \frac{1}{1 + y_1 \Delta_1}$$

- ► Estimate forward rates using the equation below

$$r_t = -\frac{\Delta \ln P_t}{\Delta t}$$

- ► Estimate deterministic drift parameter using the equation below

$$\vartheta_t = \frac{\Delta r_t}{\Delta t}$$

- ► Compare the deterministic drifts to the drift term structure estimated by the stochastic model with $\sigma \to 0$. The equations for the drift parameter in Hull-White Vasicek and CIR models in the limit $\sigma \to 0$ case are summarized as follows.

(ii) Bullet price

5.4 Validation using DerivaGem tool

DerivaGem tool is described in detail in the Appendix E. Validation of the interest rate option estimation results using the DerivaGem tool is illustrated in the Appendix F.4.1.2.

Appendix A Zero-Coupon Bond Prices and Yield Term Structure

Price of any interest rate contingent claim (denoted as f) must satisfy the following differential equation.

(A.1) $$f_t + (\phi(t,r) - \alpha r) \times f_r + \frac{1}{2}\sigma^2(t,r) \times f_{rr} - rf = 0$$

where $\phi(t,r) = \vartheta(t) - \lambda(t)\sigma(t,r)$ and $\lambda(t)$ is the market price of interest rate risk. For simplicity, we assume that

$$\lambda(t) = \lambda = 0$$

If f represents the price of zero-coupon bond, then under the affine terms structure interest arte model, the price of a zero-coupon bond is assumed to be described by the following equation:

$$f = A(t,T)e^{-B(t,T)\times r}$$

At $t = T$ the zero-coupon bond has par value, $f = 1$. Therefore, the boundary conditions are described by the following equations

(A.2) $$A(T,T) = 1 \text{ and } B(T,T) = 0$$

For the price function of the form $f = A(t,T)e^{-B(t,T)\times r}$, the following properties hold.

$$f_t = \frac{A_t}{A} \times f - rB_t \times f = \left(\frac{A_t}{A} - rB_t\right) \times f$$

$$f_r = -B \times f$$

and

$$f_{rr} = B \times f$$

After substituting the above equations in the equation (A.1), we obtain

(A.3) $$\left(\frac{A_t}{A} - rB_t\right) - (\phi(t,r) - \alpha r) \times B + \frac{1}{2}\sigma^2(t,r) \times B^2 - r = 0$$

A.1 Vasicek and Hull-White (extended Vasicek)

In this section, zero-coupon bond prices are derived for the Vasicek and Hull-White (extended Vasicek) models.

A.1.1 Hull-White (extended Vasicek)

A.1.1.1 General case

In the case of Hull-White (extended Vasicek) model the equation becomes

$$f_t + (\vartheta(t) + \alpha b - \lambda\sigma - \alpha r) \times f_r + \frac{1}{2}\sigma^2 \times f_{rr} - rf = 0$$

or equivalently, assuming

$$b = 0,\ \phi(t,r) = \vartheta(t) - \lambda\sigma \text{ and } \sigma^2(t,r) = \sigma^2$$

the equation (A.3) can be presented then as follows:

$$-r \times [B_t - \alpha B + 1] + \left[\frac{A_t}{A} - (\vartheta(t) + \lambda\sigma) \times B + \frac{1}{2}\sigma^2 \times B^2\right] = 0$$

which can be equivalently represented by the following system of equations

$$\begin{cases} B_t - \alpha B + 1 = 0 \\ \dfrac{A_t}{A} - (\vartheta(t) - \lambda\sigma) \times B + \dfrac{1}{2}\sigma^2 \times B^2 = 0 \end{cases}$$

with the boundary conditions described by equations (A.2).

The system of equations has the following solution:

$$\begin{cases} B(t,T) = \dfrac{1 - e^{-\alpha(T-t)}}{\alpha} \\ A(t,T) = e^{-\int_t^T \left[(\vartheta(s) - \lambda\sigma) \times B(s,T) - \frac{1}{2}\sigma^2 \times B^2(s,T)\right] \times ds} \end{cases}$$

or equivalently[22,23,24]

(A.4)
$$\begin{cases} B(t,T) = \dfrac{1 - e^{-\alpha(T-t)}}{\alpha} \\ A(t,T) = e^{-\left[\int_t^T [\vartheta(s) - \lambda\sigma] \times B(s,T) \times ds - \frac{\sigma^2}{2\alpha^2} \times (T - t - B(t,T)) + \frac{\sigma^2 B(t,T)^2}{4\alpha}\right]} \end{cases}$$

For $t = 0$, the equations can be represented as follows.

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22 $\int_t^T B(s,T)ds = \int_t^T \frac{1-e^{-\alpha(T-s)}}{\alpha} ds = \frac{T-t}{a} - \frac{e^{-\alpha(T-s)}}{\alpha^2}\Big|_t^T = \frac{T-t}{a} - \left(\frac{1-e^{-\alpha(T-t)}}{\alpha^2}\right) = \frac{T-t-B(t,T)}{a}$. More generally, $\int_{t_i}^{t_{i+1}} B(s,T)ds = \frac{t_{i+1}-t_i}{a} - \frac{e^{-\alpha(T-s)}}{\alpha^2}\Big|_{t_i}^{t_{i+1}} = \frac{(t_{i+1}-t_i)-(B(t_i,T)-B(t_{i+1},T))}{a}$.

23 $\int_t^T B^2(s,T)ds = \int_t^T \frac{1-2e^{-\alpha(T-s)}+e^{-2\alpha(T-s)}}{\alpha^2} ds = \frac{T-t}{\alpha^2} - 2\frac{e^{-\alpha(T-s)}}{\alpha^3}\Big|_t^T + \frac{e^{-2\alpha(T-s)}}{2\alpha^3}\Big|_t^T = \frac{T-t}{\alpha^2} - 2 \times \frac{1-e^{-\alpha(T-t)}}{\alpha^3} + \frac{1-e^{-2\alpha(T-t)}}{2\alpha^3} = \frac{T-t-B(t,T)}{\alpha^2} - \frac{B^2(t,T)}{2\alpha}$

24 $-(\lambda\sigma) \times \int_t^T B(s,T)ds - \frac{1}{2}\sigma^2 \times \int_t^T B^2(s,T)ds = \frac{\sigma^2}{\alpha^2} \times (T-t-B) \times \left(-\frac{\lambda\alpha}{\sigma}\right) - \frac{1}{2}\sigma^2\left[\frac{T-t-B}{\alpha^2} - \frac{B^2}{2\alpha}\right] = -\frac{\sigma^2}{\alpha^2} \times (T-t-B) \times \left(\frac{\lambda\alpha}{\sigma} + \frac{1}{2}\right) + \frac{\sigma^2 B^2}{4\alpha}$

(A.5)

$$\begin{cases} B(T) = \dfrac{1 - e^{-\alpha T}}{\alpha} \\ A(T) = e^{-\left[\int_0^T [\vartheta(s) - \lambda\sigma] \times B(s) \times ds - \frac{\sigma^2}{2\alpha^2} \times (T - B) + \frac{\sigma^2 B^2}{4\alpha}\right]} \end{cases}$$

If $\vartheta(s)$ is a stepwise function, then function $A(T)$ can be estimated recursively as follows. Function $A(T)$ is represented equivalently as follows

(A.6)

$$A(T) = e^{-H(T)} \times e^{\left[\frac{\sigma^2}{2\alpha^2} \times (T - B(T)) - \frac{\sigma^2 B(T)^2}{4\alpha}\right]}$$

where

$$H(T) = \int_0^T [\vartheta(s) - \lambda\sigma] \times B(s,T) \times ds$$

Suppose that $H(t_i)$ has been estimated and $\vartheta(s)$ is constant on the interval $[t_i, t_{i+1}]$ and is equal to $\vartheta(s) = \vartheta_i$. Then $H(t_{i+1})$ is estimated as follows.

(A.7)

$$H(T) = \sum_{i:t_i \leq T} (\vartheta_{i+1} - \lambda\sigma) \times \frac{(t_{i+1} - t_i) - \left(B(t_i, T) - B(t_{i+1}, T)\right)}{\alpha}$$

Values of function $A(T)$ are estimated recursively for a stepwise constant function $\vartheta(s)$ using equations (A.6) and (A.7).

A.1.1.2 Zero mean-reversion

The formulas for the special zero mean-reversion model case are derived by taking the limit $\alpha \to 0$ in the equations (A.4).[25,26]

(A.8)

$$\begin{cases} B(t, T, \alpha = 0) = T - t \\ A(t, T, \alpha = 0) = e^{-\left[\int_t^T [\vartheta(s) - \lambda\sigma] \times (T - s) ds - \frac{\sigma^2}{6} \times (T - t)^3\right]} \end{cases}$$

If $\vartheta(s)$ is a stepwise function, then function $A(T)$ can be estimated recursively as follows. Function $A(T)$ is represented equivalently as follows

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[25] $B(T-t) = \frac{1 - e^{-\alpha(T-t)}}{\alpha} \sim \frac{1 - \left(1 - \alpha(T-t) + \frac{1}{2}\alpha^2 \times (T-t)^2 - \frac{1}{6}\alpha^3 \times (T-t)^3\right)}{\alpha} = (T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\alpha^2(T-t)^3$ and therefore $\frac{(t_{i+1} - t_i) - \left(B(t_i, T) - B(t_{i+1}, T)\right)}{a} \to \frac{1}{2} \times \left((T - t_i)^2 - (T - t_{i+1})^2\right) = \left(T - \frac{t_i + t_{i+1}}{2}\right) \times (t_{i+1} - t_i)$

[26] $-\frac{\sigma^2}{\alpha^2} \times (T - t - B) \times \left(\frac{\lambda a}{\sigma} + \frac{1}{2}\right) + \frac{\sigma^2 B^2}{4\alpha} = -\frac{\sigma^2}{\alpha^2} \times \left(\frac{1}{2}\alpha(T-t)^2 - \frac{1}{6}\alpha^2(T-t)^3\right) \times \left(\frac{\lambda a}{\sigma} + \frac{1}{2}\right) + \frac{\sigma^2}{4\alpha} \times [(T-t)^2 - \alpha(T-t)^3] = -\frac{\lambda\sigma}{2} \times (T-t)^2 - \frac{\sigma^2}{6} \times (T-t)^3$

(A.9)
$$A(T) = e^{-H(T)} \times e^{\frac{\sigma^2}{6} \times (T-t)^3}$$

where

$$H(T) = \int_0^T [\vartheta(s) - \lambda\sigma] \times (T - s) \times ds$$

Suppose that $H(t_i)$ has been estimated and $\vartheta(s)$ is constant on the interval $[t_i, t_{i+1}]$ and is equal to $\vartheta(s) = \vartheta_i$. Then $H(t_{i+1})$ is estimated as follows.

(A.10)
$$H(T) = \sum_{i:t_i \leq T} (\vartheta_{i+1} - \lambda\sigma) \times \left(T - \frac{t_i + t_{i+1}}{2}\right) \times (t_{i+1} - t_i)$$

Values of function $A(T)$ are estimated recursively for a stepwise constant function $\vartheta(s)$ using equations (A.9) and (A.10).

A.1.2 Vasicek

Vasicek is a special case of the Hull-White (extended Vasicek) model with constant parameter $\vartheta(t) = \vartheta$. The formulas for the zero-coupon bond prices are simplified in this case as follows.

A.1.2.1 General case

After integrating the $\int_t^T \vartheta(s) \times B(s) \times ds$ expression[27], we get the following formula

(A.11)
$$\begin{cases} B(t,T) = \dfrac{1 - e^{-\alpha(T-t)}}{\alpha} \\ A(t,T) = e^{-\left[\left(\vartheta - \lambda\sigma - \frac{\sigma^2}{2\alpha}\right) \times \frac{(T-t-B(t,T))}{\alpha} + \frac{\sigma^2 B(t,T)^2}{4\alpha}\right]} \end{cases}$$

A.1.2.2 Zero mean-reversion

If we take the limit $\alpha \to 0$ in the equations (A.5), we get the following formulas for zero-coupon bond prices in Vasicek model with zero mean-reversion parameter

(A.12)
$$\begin{cases} B(t,T,\alpha = 0) = T - t \\ A(t,T,\alpha = 0) = e^{-\left[(\vartheta - \lambda\sigma) \times \frac{(T-t)^2}{2} - \frac{\sigma^2}{6} \times (T-t)^3\right]} \end{cases}$$

[27] $\int_t^T \vartheta(s) \times B(s) \times ds = \vartheta \times \frac{T-t-B}{a}$

A.2 CIR and Hull-White (extended CIR)

In this section, zero-coupon bond prices are derived for the CIR and Hull-White (extended CIR) models.

5.4.1 Hull-White (extended CIR)

5.4.1.1 General case

In the case of Hull-White (extended CIR) model the equation becomes

$$f_t + (\vartheta(t) + \alpha b - (\alpha + \lambda\sigma) \times r) \times f_r + \frac{1}{2}\sigma^2 \times r \times f_{rr} - rf = 0$$

or equivalently, assuming

$$b = 0,\ \psi = \alpha + \lambda\sigma, \text{ and } \sigma^2(t,r) = \sigma^2 r$$

the equation (A.3) can be presented then as follows:

$$-r \times \left[B_t - \psi B - \frac{1}{2}\sigma^2 \times B^2 + 1\right] + \left[\frac{A_t}{A} - \vartheta(t) \times B\right] = 0$$

which can be equivalently represented by the following system of equations

$$\begin{cases} B_t - \psi B - \frac{1}{2}\sigma^2 \times B^2 + 1 = 0 \\ \frac{A_t}{A} - \vartheta(t) \times B = 0 \end{cases}$$

with the boundary conditions described by equations (A.2).

The system of equations has the following solution:

(A.13)
$$\begin{cases} B(t,T) = \dfrac{2 \times \left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \alpha + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma} \\ A(t,T) = e^{-\int_t^T \vartheta(s) \times B(s,T) \times ds} \end{cases}$$

where $\gamma = \sqrt{(\alpha + \lambda\sigma)^2 + 2\sigma^2}$.

For $t = 0$, the equations can be represented as follows.

(A.14)
$$\begin{cases} B(T) = \dfrac{2 \times (e^{\gamma T} - 1)}{(\gamma + \alpha + \lambda\sigma) \times (e^{\gamma T} - 1) + 2\gamma} \\ A(T) = e^{-\int_0^T \vartheta(s) \times B(s,T) \times ds} \end{cases}$$

If $\vartheta(s)$ is a stepwise function, then function $A(T)$ can be estimated recursively as follows. Function $A(T)$ is represented equivalently as follows

(A.15) $$A(T) = e^{-H(T)}$$

where

$$H(T) = \int_0^T \vartheta(s) \times B(s,T) \times ds$$

Suppose that $H(t_i)$ has been estimated and $\vartheta(s)$ is constant on the interval $[t_i, t_{i+1}]$ and is equal to $\vartheta(s) = \vartheta_i$. Then $H(t_{i+1})$ is estimated as follows.

(A.16)
$$H(T) = \sum_{i:t_i \le T} \vartheta_i \times \Big[-\frac{2}{\gamma - \alpha} \times [t_{i+1} - t_i] + \frac{4}{\gamma^2 - \alpha^2} \times \left(\ln[(\gamma + \alpha) \times e^{\gamma t_{i+1}} + (\gamma - \alpha)] - \ln[(\gamma + \alpha) \times e^{\gamma t_i} + (\gamma - \alpha)] \right) \Big]$$

Values of function $A(T)$ are estimated recursively for a stepwise constant function $\vartheta(s)$ using equations (A.15) and (A.16).

5.4.1.2 Zero mean-reversion

The formulas for the special zero mean-reversion model case are derived by taking the limit $\alpha \to 0$ in the equations (A.13). As $\alpha \to 0$, γ converges to $\gamma = \sigma\sqrt{2 + \lambda^2}$ and the equations (A.13) can be represented as follows.

(A.17)
$$\begin{cases} B(t, T, \alpha = 0) = \dfrac{2 \times \left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma} \\ A(t, T, \alpha = 0) = e^{-\int_t^T \vartheta(s) \times B(s) \times ds} \end{cases}$$

With zero mean-reversion parameter equation (A.16) is described as follows.

(A.18)
$$H(T) = \sum_{i:t_i \le T} \vartheta_i \times \left[-\frac{2}{\gamma} \times [t_{i+1} - t_i] + \frac{4}{\gamma^2} \times \left(\ln[e^{\gamma t_{i+1}} + 1] - \ln[e^{\gamma t_i} + 1] \right) \right]$$

A.2.1 CIR

CIR is a special case of the Hull-White (extended CIR) model with constant parameter $\vartheta(t) = \vartheta$. The formulas for the zero-coupon bond prices are simplified in this case as follows.

A.2.1.1 General case

After integrating the $\int_t^T \vartheta(s) \times B(s) \times ds$ expression, we get the following formula

(A.19)

$$\begin{cases} B(t,T) = \dfrac{2 \times \left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma} \\ A(t,T) = \left[\dfrac{2 \times \gamma \times e^{(\gamma + \alpha + \lambda\sigma)(T-t)/2}}{(\gamma + a + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2\vartheta}{\sigma^2}} \end{cases}$$

A.2.1.2 Zero mean-reversion

If we take the limit $\alpha \to 0$ in the equations (A.19), we get the following formulas for zero-coupon bond prices in CIR model with zero mean-reversion parameter

(A.20)

$$\begin{cases} B(t,T,\alpha = 0) = \dfrac{2 \times \left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma} \\ A(t,T,\alpha = 0) = \left[\dfrac{2 \times \gamma \times e^{(\gamma + \lambda\sigma)(T-t)/2}}{(\gamma + \lambda\sigma) \times (e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{\frac{2\vartheta}{\sigma^2}} \end{cases}$$

Appendix B Interest Rate Distribution Parameters

The derived mean and variance parameters of the interest rate distribution are applied to validate the numerically calculated distribution of the interest rates. The mean and variance are estimated based on the interest rate model

$$dr_t = (\vartheta(t) - ar_t) \times dt + \sigma(t, r_t) \times dW_t$$

which can be equivalently represented as

$$r_{t+dt} = r_t \times (1 - \alpha dt) + \sigma(t, r_t) \times dW_t + \vartheta(t)dt$$

B.1 Distribution mean

Suppose that $\mu_t = Er_t$. Then the differential equation for the μ_t function is described by the following formula.

$$\mu_{t+dt} = \mu_t \times (1 - \alpha dt) + \vartheta(t)dt$$

or equivalently

$$\mu' + \alpha\mu = \vartheta(t)$$

B.1.1 Stepwise-constant drift

Suppose that $\mu = e^{-\alpha t} \times \eta$. Then $e^{-\alpha t} \times \eta' = \vartheta(t)$ or $\eta = r + \int_0^t e^{\alpha s}\,\vartheta(s)ds$. Therefore, we get the following formula in general case.

(B.1) $$\mu = e^{-\alpha t} \times \left[r + \int_0^t e^{\alpha s}\,\vartheta(s)ds \right]$$

For a stepwise constant $\vartheta(u)$ function, equation (B.1) can be represented as follows[28]

(B.2) $$\mu = e^{-\alpha t} \times \left[r + \frac{1}{\alpha} \times \sum_{i:t_i<t} (e^{at_{i+1}} - e^{at_i}) \times \vartheta_i \right]$$

B.1.2 Stepwise-constant drift and zero mean-reversion

If $\alpha \to 0$ then the general equation (B.2) is simplified to the following equation

[28] $\int_{t_i}^{t_{i+1}} e^{\alpha s} ds = \frac{1}{\alpha} \times (e^{at_{i+1}} - e^{at_i})$

$$\mu = r + \int_0^t \vartheta(s)ds = r + \sum_{i:t_i<t} (t_{i+1} - t_i) \times \vartheta_i \tag{B.3}$$

B.1.3 Constant drift

For the constant $\vartheta(t) = \vartheta$ parameter equation (B.2) is reduced to the following formula

$$\mu = \frac{\vartheta}{\alpha} + e^{-\alpha t} \times \left(r - \frac{\vartheta}{\alpha}\right) \tag{B.4}$$

Note that $\frac{\vartheta}{\alpha}$ is the long-term equilibrium of the interest rates.

B.1.4 Constant drift and zero mean-reversion

For the constant $\vartheta(t) = \vartheta$ parameter and $\alpha \to 0$ equation (B.4) is reduced to the following formula

$$\mu = r + \vartheta t \tag{B.5}$$

Note that the equations derived for the interest rate distribution mean parameter apply to both Vasicek (Hull-White extended Vasicek) and to CIR (Hull-White extended CIR) interest rate models.

B.2 Distribution variance

The equation for the interest rate variance is derived separately for the Vasicek and CIR models.

B.2.1 Vasicek and Hull-White (extended Vasicek)

In the case of Hull-White (extended Vasicek) model, the variance of the interest rates is derived based on the following equation.

$$r_{t+dt} = r_t \times (1 - \alpha dt) + \sigma \times dW_t + \vartheta(t)dt$$

Suppose that ν_t denotes the variance of the interest rates. Then the variance ν_t is described by the following equation

$$\nu_{t+dt} = \nu_t \times (1 - \alpha dt)^2 + \sigma^2 dt$$

The equation can be represented equivalently as follows

$$\nu_t' + 2\alpha \nu_t = \sigma^2$$

If we look for the solution in the form $\nu_t = e^{-2\alpha t} \times \upsilon_t$, then $e^{-2\alpha t} \times \upsilon_t' = \sigma^2$ or $\upsilon_t = \frac{\sigma^2}{2\alpha} \times (e^{2\alpha t} - 1)$. The equation for ν_t then becomes

(B.6) $$\nu_t = \frac{\sigma^2}{2\alpha} \times (1 - e^{-2\alpha t})$$

As $\boldsymbol{\alpha} \to \mathbf{0}$, equation (B.6) becomes

(B.7) $$\nu_t = \sigma^2 \times t$$

B.2.2 Hull-White (extended CIR)

In the case of Hull-White (extended Vasicek) model, the variance of the interest rates is derived based on the following equation.

$$r_{t+dt} = r_t \times (1 - \alpha dt) + \sigma\sqrt{r_t} \times dW_t + \vartheta(t)dt$$

Suppose that ν_t denotes the variance of the interest rates. Then the variance ν_t is described by the following equation.

$$\nu_{t+dt} = E\left[r_t \times (1 - \alpha dt) + \sigma\sqrt{r_t} \times dW_t\right]^2 - \left(E\left[r_t \times (1 - \alpha dt) + \sigma\sqrt{r_t} \times dW_t\right]\right)^2$$

or equivalently

$$\nu_{t+dt} = E[r_t \times (1 - \alpha dt)]^2 + \sigma^2 \times \mu_t \times dt - (E[r_t \times (1 - \alpha dt)])^2 = \nu_t \times (1 - \alpha dt)^2 + \sigma^2 \times \mu_t \times dt$$

The above formula can be represented as the following differential equation

$$\nu_t' + 2\alpha\nu_t = \sigma^2 \times \mu_t$$

where $\mu_t = e^{-\alpha t} \times \left[r + \int_0^t e^{\alpha s}\, \vartheta(s)ds\right]$ was derived in equation (B.1).

The generic solution of the differential equation is described by the following formula[29]

(B.8) $$\nu_t = \frac{\sigma^2 \times e^{-2\alpha t}}{\alpha} \times \left[r \times (e^{\alpha t} - 1) + \int_0^t e^{au} \times (e^{at} - e^{au}) \times \vartheta_u du\right]$$

For a stepwise constant $\vartheta(u)$ function, equation (B.8) can be represented as follows.[30]

[29] $\nu_t = \sigma^2 \times e^{-2\alpha t} \times \int_0^t e^{2\alpha s}\mu_s ds = \sigma^2 \times e^{-2\alpha t} \times \int_0^t e^{\alpha s} \times [r + \int_0^s e^{au}\vartheta_u du]ds = \sigma^2 \times e^{-2\alpha t} \times \left[r \times \int_0^t e^{\alpha s} + \int_0^t e^{au}\vartheta_u \times \left(\int_u^t e^{\alpha s}ds\right) \times du\right] = \frac{\sigma^2 \times e^{-2\alpha t}}{\alpha} \times \left[r \times (e^{\alpha t} - 1) + \int_0^t e^{au} \times (e^{at} - e^{au}) \times \vartheta_u\right]$

[30] $\int_{t_i}^{t_{i+1}} e^{au}(e^{at} - e^{au}) \times du = \frac{1}{\alpha} \times \left[e^{at} \times (e^{at_{i+1}} - e^{at_i}) - \frac{1}{2} \times (e^{2at_{i+1}} - e^{2at_i})\right] = \frac{1}{\alpha} \times (e^{at_{i+1}} - e^{at_i}) \times \left(e^{at} - \frac{e^{at_{i+1}} + e^{at_i}}{2}\right)$

(B.9) $$\nu_t = \frac{\sigma^2 \times e^{-2\alpha t}}{\alpha} \times \left[r \times (e^{\alpha t} - 1) + \frac{1}{\alpha} \times \sum_{i:t_i<t} (e^{\alpha t_{i+1}} - e^{\alpha t_i}) \times \left(e^{\alpha t} - \frac{e^{\alpha t_{i+1}} + e^{\alpha t_i}}{2} \right) \times \vartheta_i \right]$$

As $\boldsymbol{\alpha} \to \mathbf{0}$, equation (B.8) becomes

(B.10) $$\nu_t = \sigma^2 \times \left[rt + \int_0^t (t-u) \times \vartheta(u) du \right]$$

For a stepwise constant $\vartheta(u)$ function, equation (B.10) can be represented as follows[31]

(B.11) $$\nu_t = \sigma^2 \times \left[rt + \sum_{i:t_i<t} (t_{i+1} - t_i) \times \left(t - \frac{t_{i+1} + t_i}{2} \right) \times \vartheta_i \right]$$

B.2.3 CIR

If parameter $\vartheta(t) = \vartheta$ is constant, the equation (B.8) can be simplified as follows.

(B.12) $$\nu_t = \sigma^2 \times \left[\frac{r}{\alpha} \times e^{-\alpha t} \times (1 - e^{-\alpha t}) + \frac{\vartheta}{\alpha^2} \times \left(\frac{1}{2} - e^{-\alpha t} + \frac{1}{2} e^{-2\alpha t} \right) \right]$$

As $\boldsymbol{\alpha} \to \mathbf{0}$, equation (B.12) becomes

(B.13) $$\nu_t = \sigma^2 \times \left[rt + \vartheta \times \frac{t^2}{2} \right]$$

B.3 Transition probabilities

In this section we consider two alternative approaches to adjusting transition probabilities. Under the first (default) approach, matching with the theoretical mean and standard deviation parameters of the interest rate process is performed by applying a contraction mapping to the mean and standard deviation parameters. The approach is described in Appendix B.3.1.

In an alternative approach, transition probability adjustment is performed by making a minimum adjustment to the pre-adjusted probabilities so that the mean and standard deviation of the adjusted probabilities match the theoretical mean and standard deviation parameters. A potential problem with the approach is that some of the adjusted transition probabilities may be negative. If we impose a restriction that the adjusted probabilities are non-negative numbers, then implementation of the approach may become more time intensive and inefficient compared to the first approach. The approach is described in Appendix B.3.2.

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[31] $\int_{t_i}^{t_{i+1}} (t-u)du = -\frac{(t-u)^2}{2}\Big|_{t_i}^{t_{i+1}} = \frac{(t-t_i)^2}{2} - \frac{(t-t_{i+1})^2}{2} = \frac{2 \times t \times (t_{i+1}-t_i) - (t_{i+1}^2 - t_i^2)}{2} = (t_{i+1} - t_i) \times \left(t - \frac{t_{i+1}+t_i}{2} \right)$

B.3.1 Adjustment estimation based on contraction mapping

Suppose that μ and σ are the parameters of the transition probabilities that we need to match in state $r_{t,i}$ and that $\hat{\mu}(\mu_j, \sigma_j)$ and $\hat{\sigma}(\mu_j, \sigma_j)$ are the mean and standard deviation parameters of transition probabilities estimated numerically in state $r_{t,i}$ based on the discrete approximation of the process set of states. Then the contraction mapping is defined as follows.

$$\begin{cases} \mu_{j+1} = \mu_j + \lambda_\mu \times \left[\mu - \hat{\mu}(\mu_j, \sigma_j)\right] \\ \sigma_{j+1} = \sigma_j + \lambda_\sigma \times \left[\sigma - \hat{\sigma}(\mu_j, \sigma_j)\right] \end{cases}$$

for some parameters $\lambda_\mu \leq 1$ amd $\lambda_\sigma \leq 1$. If the above mapping is a contraction mapping [Note: need to prove it] then the sequence of parameters (μ_j, σ_j) converges to a solution

$$\begin{cases} \mu_{j+1} = \mu_j \\ \sigma_{j+1} = \sigma_j \end{cases}$$

or equivalently

$$\begin{cases} \hat{\mu}(\mu_j, \sigma_j) = \mu \\ \hat{\sigma}(\mu_j, \sigma_j) = \sigma \end{cases}$$

The algorithm can be summarized as follows.

- Start with the theoretical values $\mu_0 = \mu$ and $\sigma_0 = \sigma$;
- At each iteration j, estimate the transition distribution probabilities assuming Normal distribution with parameters (μ_j, σ_j);
- Estimate numeric mean and standard deviation parameters $(\mu_{j+1}, \sigma_{j+1})$ based on the discrete distribution estimated at the previous step;
- If the numeric mean $\hat{\mu}(\mu_j, \sigma_j)$ (standard deviation $\hat{\sigma}(\mu_j, \sigma_j)$) is below (above) the theoretical value at iteration j, then we respectively increase (decrease) the mean (standard deviation) parameter;
- Continue the iterations until the numerical value converges to the theoretical value sufficiently close.
- Parameters λ_μ and λ_σ are selected to ensure that the mapping is a contraction mapping and to maximize the speed of convergence to the theoretical mean and standard deviation values.

B.3.2 Adjustment estimation based on minimum deviation from pre-adjusted normal probabilities

As discussed in Section 4.2, the transition probabilities are estimated by solving the following optimization problem. In the notation below we assume that period t and process state $r_{t,i}$ are fixed and that the transition probabilities in the state $r_{t,i}$ are estimated (we omit the indices t and i in the equation below).

$$\min \frac{1}{2} \sum_j q_j^{-\rho} \times \left(\tilde{q}_j - q_j\right)^2$$

under the following constraints:

$$\begin{cases} \sum_j \tilde{q}_j = 1 \\ \sum_j \tilde{q}_j \times \left(r_{t+dt,j} - r_{t,i}\right) = \mu\left(t, r_{t,i}\right) \times dt = m \\ \sum_j \tilde{q}_j \times \left(r_{t+dt,j} - r_{t,i}\right)^2 = \sigma^2(t, r_t) \times dt + [\mu(t, r_t) \times dt]^2 = s_2 \end{cases}$$

Note that because the set of constraints consist of three equations, the number of branches in the modelling tree must be at least three to ensure that the system of constraints has a solution. It is generally not possible to match the drift and volatility parameters of an interest rate process using binary trees to approximate the Brownian motion.

Lagrangian of the optimization model is described by the following equation

$$\mathcal{L} = -\frac{1}{2}\sum_j \left(\tilde{q}_j - q_j\right)^2 + \lambda_1 \times \left[1 - \sum_j \tilde{q}_j\right] + \lambda_2 \times \left[m - \sum_j \tilde{q}_j \times \left(r_{t+dt,j} - r_{t,i}\right)\right] + \lambda_3 \times \left[s_2 - \sum_j \tilde{q}_j \times \left(r_{t+dt,j} - r_{t,i}\right)^2\right]$$

In the matrix notation, the Lagrangian can be represented as follows.

$$\mathcal{L} = -\frac{1}{2}(\tilde{q} - q)^T \times D \times (\tilde{q} - q) - \tilde{q}^T \times X \times \lambda + (\lambda_1 + \lambda_2 \times m + \lambda_3 s_2)$$

where

$$D = \begin{pmatrix} q_1^{-\rho} & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & q_n^{-\rho} \end{pmatrix} \text{ and } X = \begin{pmatrix} 1 & r_{t+dt,1} - r_{t,i} & \left(r_{t+dt,1} - r_{t,i}\right)^2 \\ \cdots & \cdots & \cdots \\ 1 & r_{t+dt,n} - r_{t,i} & \left(r_{t+dt,n} - r_{t,i}\right)^2 \end{pmatrix}$$

The first-order conditions for the optimization problem are described by the following equations:

$$\begin{cases} D \times (\tilde{q} - q) = X \times \lambda \\ X^T \times \tilde{q} = c \end{cases}$$

where $c = (1, m, s_2)$. The system of equations has the following solution

$$\begin{cases} \lambda = (X^T D^{-1} X)^{-1} \times (c - X^T q) \\ \tilde{q} = q + D^{-1} \times X \times \lambda \end{cases}$$

The above equation does not guarantee that estimated adjusted probabilities $\tilde{q}$ are positive. The negative values of $\tilde{q}$ are replaced with zeros and the probabilities are normalized. As a result, there may be some deviation of the mean and standard deviation parameters from the theoretical values. Examples are illustrated in the Appendix G.2.

Appendix C Alternative Parameter Estimation Methods

Section 3 describes the default approach to the option model parameters estimation (which is implemented in the ac.finance.SRM tool. This section summarizes other alternative parameter estimation methods.

C.1 Volatility parameter

Alternative methods for volatility parameter estimation are summarized below.

C.1.1 Sample variance based on daily data

The sample volatility is derived for Vasicek and CIR models. Volatility is estimated for the daily frequency and then is converted into the annual volatility using the following equation:[32]

(C.1)
$$\hat{\sigma} = \sqrt{250} \times \sigma_n$$

where u_t is the error term of the interest rate process. The equation for the error term u_t is derived separately for each type of the interest rate process. The key downside of the approach is the sensitivity of the volatility estimator to the sample outliers (which impact is amplified by the $\frac{1}{\sqrt{dt}}$ factor).

C.1.1.1 Vasicek model

Sample volatility is estimated based on the following short rate representation.

$$dr_t = (\vartheta_t - ar_t) \times dt + \sigma \times dW_t$$

so that the residuals are estimated as follows

(C.2)
$$u_t = \frac{dr_t}{\sqrt{dt}}$$

In practical applications, dt is assumed constant and is calculated based on the number of business days during the year: $dt = \frac{1}{250}$. Equation **Error! Reference source not found.** represents a daily sample of residuals, which can be used to estimate the volatility parameter. A potential problem with the daily sample is that outlier data can have a material impact on the volatility estimation.

C.1.1.2 CIR and Hull-White (extended CIR)

Sample volatility is estimated based on the following short rate representation.

$$dr_t = (\vartheta_t - ar_t) \times dt + \sigma \times \sqrt{r_t} \times dW_t$$

so that $\hat{\sigma}$ is estimated as a sample standard deviation of the $\left\{\frac{dr_t}{\sqrt{r_t \times dt}}\right\}$ sample:

[32] The equation assumes that the yield series has daily frequency. If the frequency of the yield series is different, the equation must be adjusted accordingly.

(C.3) $$u_t = \frac{dr_t}{\sqrt{r_t \times dt}}$$

In practical applications, dt is assumed constant and is calculated based on the number of business days during the year: $dt = \frac{1}{250}$.

C.1.2 EWMA / GARCH(1, 1) variance estimate

Under the GARCH(1, 1) approach, the volatility is modelled as follows:

(C.4) $$\begin{cases} u_n = \sigma_n \varepsilon_n \\ \sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \end{cases}$$

where

(C.5) $$\gamma + \alpha + \beta = 1$$

The volatility in each specific period is conditional on the previous period volatility and this period change in the interest rates. Term V_L represents long-term volatility. This is a latent variable which is calculated as part of the GARCH(1, 1) model estimation process.

The model is estimated as a regression equation using maximum likelihood methods.

(C.6) $$\begin{cases} u_n = \sigma_n \varepsilon_n \\ \sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \end{cases}$$

Parameters γ and V_L are estimated then as follows

(C.7) $$\gamma = 1 - \alpha - \beta \text{ and } V_L = \frac{\omega}{\gamma}$$

If the intercept parameter ω is estimated as negative, then it is set to zero and the GARCH(1, 1) model becomes EWMA model described by the following equation

(C.8) $$\begin{cases} u_n = \sigma_n \varepsilon_n \\ \sigma_n^2 = \alpha \varepsilon_{n-1}^2 + (1 - \alpha) \sigma_{n-1}^2 \end{cases}$$

Parameter α in JPMorgan RiskMetrics[33] tool is set to

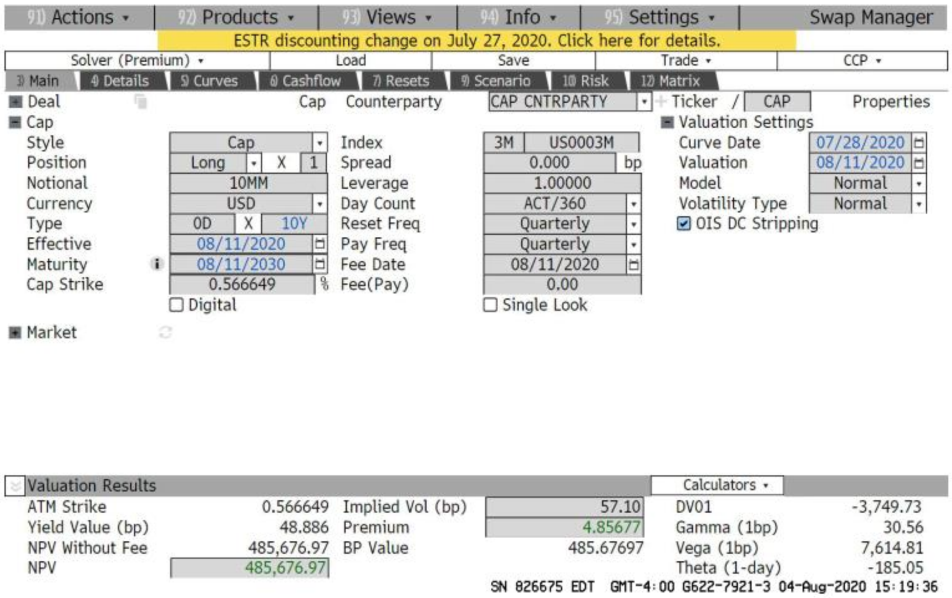
(C.9) $$\alpha = 0.06 \text{ and } 1 - \alpha = 0.94$$

based on the statistical analysis performed by JPMorgan (see [1] pages 198 – 205).

[33] https://en.wikipedia.org/wiki/RiskMetrics.

C.1.3 Volatility based on Bloomberg's SWPM tool

Bloomberg Swap Manager (**SWPM**) tool estimates implied volatility of a base rate (including volatility of Libor rates) as part of interest rate cap calculations. The screen with the output of interest rate cap calculations is provided in the exhibit below.



The exhibit above shows that, as part of the interest rate cap valuation, SWPM estimated implied Libor volatility at 0.57%.

C.1.4 CBOE interest rate volatility index

Alternatively, volatility parameter can be estimated using CBOE SRVIX interest rate volatility index[34] reported by Bloomberg and illustrated in the exhibit below. The description of the SRVIX index estimation methodology is described in the white paper released by CBOE. [35] According to the white paper, the volatility index is estimated based on the Black (1976) formula for the underlying bond prices.[36]

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[34] SRVIX index overview: http://www.cboe.com/index/dashboard/srvix#srvix-overview.

[35] SRVIX index white paper: https://www.cboe.com/micro/srvix/srvix.pdf.

[36] https://en.wikipedia.org/wiki/Black_model.

Exhibit 5.3 Interest rate volatility index SRVIX

C.2 Drift parameter

The section describes the equations for the drift parameter estimation in Hull-White (extended Vasicek) and Hull-White (extended CIR) models.

C.2.1 Hull-White (extended Vasicek) model

The term structure for the Hull-White (extended Vasicek) is described by the following equations[37]

$$TR_T = B(0,T) \times r_0 + \int_0^T \vartheta(s) \times B(s,T) \times ds - \frac{\sigma^2}{2\alpha^2} \times \big(T - B(0,T)\big) + \frac{\sigma^2 B(0,T)^2}{4\alpha}$$

which, in case $\alpha = 0$ becomes

$$TR_T = T \times r_0 + \int_0^T \vartheta(s) \times (T-s) \times ds - \frac{\sigma^2}{6} \times T^3$$

Suppose that

$$G(T) = \int_0^T \vartheta(s) \times B(s,T) \times ds = TR_T - B(0,T) \times r_0 + \frac{\sigma^2}{2\alpha^2} \times \big(T - B(0,T)\big) - \frac{\sigma^2 B(0,T)^2}{4\alpha}$$

which, in case $\alpha = 0$ becomes

$$G(T) = \int_0^T \vartheta(s) \times (T-s) \times ds = TR_T - T \times r_0 + \frac{\sigma^2}{6} \times T^3$$

Suppose that $t_0 = 0, t_1, \dots, t_{n-1}, t_n = T$ are discrete periods with observed market yield rates R_i and respective values of the function $G(T)$ equal to G_i. Function $\vartheta(s)$ is assumed to be constant and equal to ϑ_{i+1} on the interval $t \in$

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[37] A reminder that $B(s,t) = \frac{1-e^{-\alpha(T-s)}}{\alpha} \to T - s$ as $\alpha \to 0$

$[t_i, t_{i+1}]$. If we denote $\beta_{i+1,n} = \int_{t_i}^{t_{i+1}} B(s,T) \times ds$, then the unknown parameters ϑ_i are estimated recursively as described in the equation below.

$$\begin{cases} \vartheta_1 = \dfrac{G_1}{\beta_{1,1}}, & if\ n = 1 \\ \vartheta_n = \dfrac{G_n - \sum_{i=1}^{n-1} \vartheta_i \times \beta_{i,n}}{\beta_{n,n}} & if\ n > 1 \end{cases} \quad \text{(C.10)}$$

where

$$\beta_{i+1,n} = \frac{(t_{i+1} - t_i) - \big(B(T - t_i) - B(T - t_{i+1})\big)}{\alpha}$$

If $\alpha \to 0$, then the equation for $\beta_{i+1,n}$ is described as follows.

$$\beta_{i+1,n} = (t_{i+1} - t_i) \times (T - t_i^{av})$$

where $t_i^{av} = \frac{t_{i+1} + t_i}{2}$.

In the case of the Hull-White model, the discrete term structure of the interest rates is assumed to be generated by a piecewise-constant structure of drift parameters: $t_i \to \vartheta_{i+1}$ (using equation (vsk.3b)), which is then converted to a function defined as

$$\vartheta(t) = \vartheta_{i+1} \quad for \quad \vartheta \in [t_i, t_{i+1})$$

C.2.2 Hull-White (extended CIR) model

As before, suppose that $t_0 = 0, t_1, \dots, t_{n-1}, t_n = T$ are discrete periods with observed market yield rates R_i and respective values of the function $G(T)$ equal to G_i. If we denote $\beta_{i+1,n} = \int_{t_i}^{t_{i+1}} B(s,T) \times ds$, then the unknown parameters ϑ_i are estimated recursively as described in the equation below.

$$\begin{cases} \vartheta_1 = \dfrac{G_1}{\beta_{1,1}}, & if\ n = 1 \\ \vartheta_n = \dfrac{G_n - \sum_{i=1}^{n-1} \vartheta_i \times \beta_{i,n}}{\beta_{n,n}} & if\ n > 1 \end{cases} \quad \text{(C.11)}$$

where[38]

$$\beta_{i+1,n} = \int_{t_i}^{t_{i+1}} B(s,T) \times ds = \frac{-2}{\gamma - \alpha} \times [t_{i+1} - t_i] + \frac{4}{\gamma^2 - \alpha^2} \times \left(\ln\left[(\gamma + \alpha) \times e^{\gamma(T - t_i)} + (\gamma - \alpha)\right] - \ln\left[(\gamma + \alpha) \times e^{\gamma(T - t_{i+1})} + (\gamma - \alpha)\right]\right)$$

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38 $\int_{t_i}^{t_{i+1}} B(s,T) \times ds = \int_{t_i}^{t_{i+1}} \frac{2 \times \left(e^{\gamma(T-s)} - 1\right)}{(\gamma+\alpha) \times \left(e^{\gamma(T-s)} - 1\right) + 2\gamma} \times ds = \int_{t_i}^{t_{i+1}} \frac{2 \times e^{\gamma(T-s)} - 2}{(\gamma+\alpha) \times e^{\gamma(T-s)} + (\gamma-\alpha)} \times ds = \frac{-2}{\gamma-\alpha} \int_{t_i}^{t_{i+1}} \frac{\left[(\gamma+\alpha) \times e^{\gamma(T-s)} + (\gamma-\alpha)\right] - e^{\gamma(T-s)} \times \left((\gamma-\alpha) + (\gamma+\alpha)\right)}{(\gamma+\alpha) \times e^{\gamma(T-s)} + (\gamma-\alpha)} \times ds = \frac{-2}{\gamma-\alpha} \times [t_{i+1} - t_i] - \frac{2}{\gamma-\alpha} \times 2 \times \int_{t_i}^{t_{i+1}} \frac{de^{\gamma(T-s)}}{(\gamma+\alpha) \times e^{\gamma(T-s)} + (\gamma-\alpha)} = \frac{-2}{\gamma-\alpha} \times [t_{i+1} - t_i] - \frac{4}{(\gamma+\alpha)(\gamma-\alpha)} \times \ln\left[(\gamma+\alpha) \times e^{\gamma(T-s)} + (\gamma-\alpha)\right] \Big|_{t_i}^{t_{i+1}} = \frac{-2}{\gamma-\alpha} \times [t_{i+1} - t_i] + \frac{4}{\gamma^2 - \alpha^2} \times \left(\ln\left[(\gamma+\alpha) \times e^{\gamma(T-t_i)} + (\gamma-\alpha)\right] - \ln\left[(\gamma+\alpha) \times e^{\gamma(T-t_{i+1})} + (\gamma-\alpha)\right]\right)$

If the mean-reversion parameter is zero, $\alpha = 0$, then the above equation is simplified to the following equation.

$$\beta_{i+1,n} = \frac{-2}{\gamma} \times [t_{i+1} - t_i] + \frac{4}{\gamma^2} \times \left(\ln\left[e^{\gamma(T-t_i)} + 1\right] - \ln\left[e^{\gamma(T-t_{i+1})} + 1\right]\right)$$

C.3 Mean-reversion parameter

Alternative methods for mean-reversion parameter estimation are summarized below.

C.3.1 Mean-reversion parameter implied by term structure

The implied mean-reversion parameter is estimated based on the yield term structure.

C.3.1.1 Vasicek - unconstrained

After substituting $P_T = \left[e^{\frac{(B-T)\times(\alpha^2\beta-\sigma^2/2)}{\alpha^2} - \frac{\sigma^2 B^2}{4\alpha}}\right] \times e^{-Br_0}$, we get $-\ln P_T = \frac{\sigma^2 B^2}{4\alpha} - \frac{(B-T)\times\left(\alpha^2\beta - \frac{\sigma^2}{2}\right)}{\alpha^2} + Br_0$ or

$$TR_T - Br_0 = \frac{\sigma^2 B^2}{4\alpha} - \frac{(B-T)\times\left(\alpha\theta - \frac{\sigma^2}{2}\right)}{\alpha^2}$$

The equation can be rewritten as

$$TR_T - Br_0 = -\beta \times (B - T) + \sigma^2 \times \left[\frac{B^2}{4\alpha} + \frac{B-T}{2\alpha^2}\right]$$

The parameters are estimated as follows. For a given value of α, the function $B = \frac{1-e^{-\alpha T}}{\alpha}$ is estimated and the above linear model is estimated. The coefficients σ^2 and β are estimated from the linear model and parameter α is selected to minimize the overall sum of squares in the linear regression. The estimation procedure is reduced to the optimization problem for a non-linear function of α variable.

C.3.1.2 Vasicek - constrained

In the constrained version of the model, $\beta = r_0$ so that the steady state is equal to the current yield rate. The equation then becomes:

$$TR_T - Tr_0 = \sigma^2 \times \left[\frac{B^2}{4\alpha} + \frac{B-T}{2\alpha^2}\right]$$

or, formally,

$$B(t) = t, \;\; A_1 = 0, \;\; and \;\; A_2 = \frac{\tilde{B}^2}{4\alpha} + \frac{\tilde{B} - T}{2\alpha^2}$$

Where $\tilde{B} = \frac{1-e^{-\alpha T}}{\alpha}$. The price in the case of constrained model is calculated as follows

$$P_T = e^{-\sigma^2\left[\frac{\tilde{B}^2}{4\alpha} + \frac{\tilde{B}-T}{2\alpha^2}\right]} \times e^{-Tr_0}$$

Appendix D Bond Repayment Terms and Option Penalty Structure

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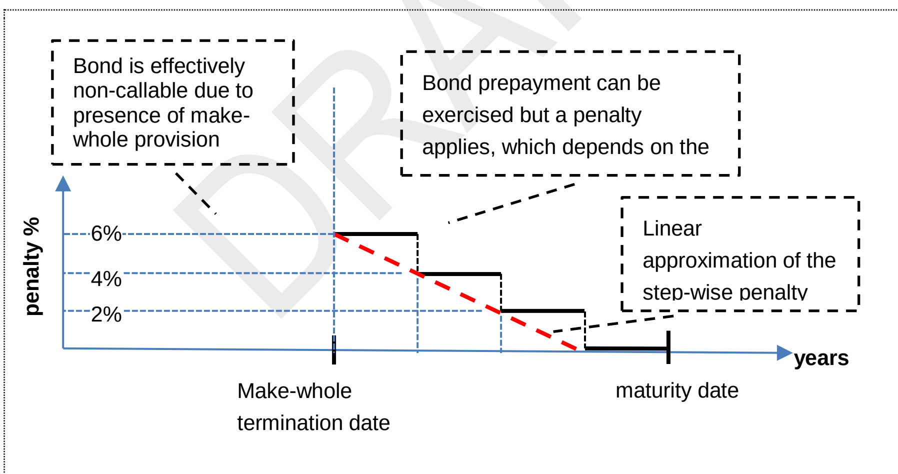
D.1 Call option

A typical penalty structure of a callable bond can be summarized as follows:

- **Make-whole provision**. The callable bond has a make-whole provision, which is effective until certain make-whole provision termination date. Prior to a make-whole termination date, a cost of exercising a call option is assumed to be very high and effectively the note is assumed to be non-callable because a size of a make-whole penalty makes it very expensive for a borrower to prepay early.
- **Redemption penalty**. On a make-whole termination date, a prepayment penalty is set as a fixed percentage of a note principal amount. This penalty rate varied for different callable bonds.
- **Redemption penalty structure**. A penalty fixed percentage is reduced over time. A penalty reduction is approximately linear over time: after every fixed period of time, a penalty is reduced by a fixed amount until a penalty becomes zero on a callable bond maturity date.

This penalty structure can be presented schematically using the following diagram:

Exhibit D.1 Callable bond prepayment penalty structure



In the example above, the post-make-whole-termination penalty is assumed to be 6% on the make-whole termination date. The penalty rate is reduced to 4% and then to 2% before the bond matures. The penalty structure is presented by the respective stepwise penalty function. The stepwise function can be approximated reasonably accurately with the linear function (presented in the diagram by the red dashed line) which decreases from 6% on the make-whole termination date to 0% on the maturity date.

Bloomberg print screen with a typical callable bond redemption penalty schedule is presented in the exhibit below.

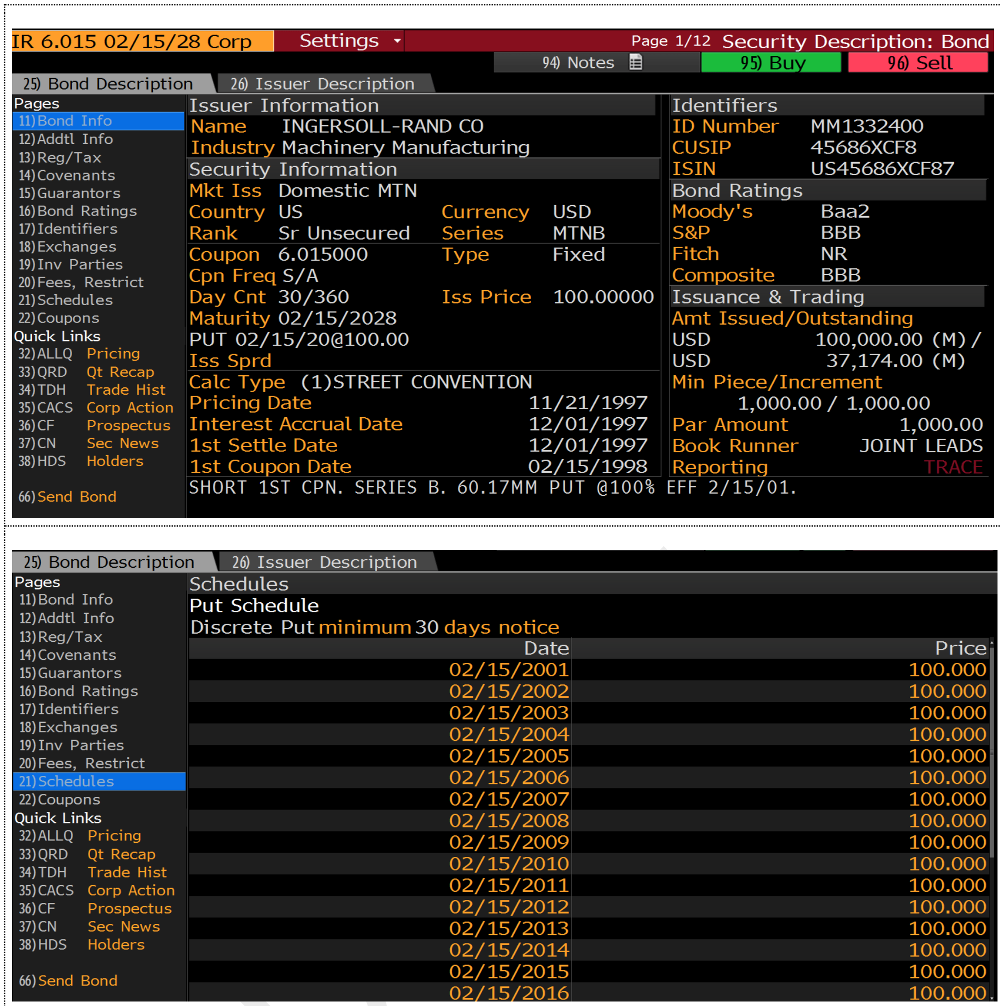
Exhibit D.2 A standard penalty structure of a callable bond

In the example, the callable bond was issued in March 2015, has a make-whole provision which terminates in March 2020 and has a redemption penalty of 2.875% applicable in March 2020 which is reduced annually to zero in March 2023 (two years prior to the bond maturity).

D.2 Put option

A putable bond can typically be exercised only at certain discrete set of dates. Bloomberg print screen with a typical putable bond redemption penalty schedule is presented in the exhibit below.

Exhibit D.3 Standard pay-on-demand terms of a putable bond



In the example, the put option can be exercised on an annual basis on a specific date (February 15) starting after approximately three years and 2.5 months after the bond issue date. In many examples of putable bonds pay-on-demand terms the put option can be exercised only once during the life of the bond. Putable bonds typically do not have redemption discounts. The main restriction on the pay-on-demand terms is the timing of exercising the option.

Appendix E John Hull DerivaGem Tool

In this section we describe the Hull-White DerivaGem tool, which is traditionally used to valuate the interest rate options. Our tool effectively replicates the numbers produced by the DerivaGem tool.[39]

E.1 Description of DerivaGem tool

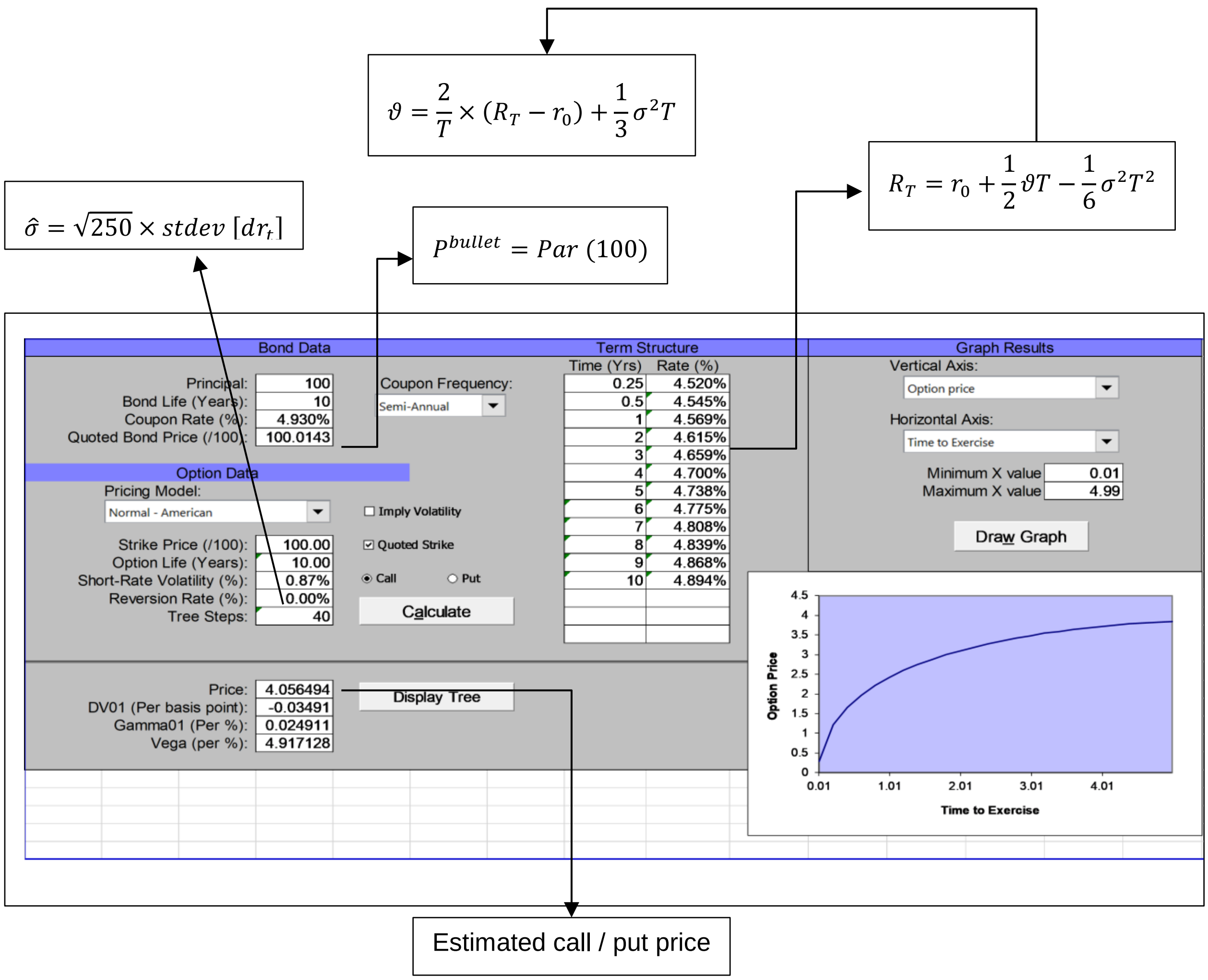
The following parameters are set to estimate the call / put option price (the parameters are illustrated for the case of Vasicek model with zero mean-reversion parameter):

1. Principal amount (value of the call option is calculated relative to the principal amount. Therefore, if the principal amount is set equal to 100, then the call option is calculated as a percentage of the principal amount);
2. Maturity term of the bond (in years);
3. Coupon rate is set equal to the bond coupon rate;
4. Strike price (redemption price of the bond). The strike price is set equal to the principal amount if there is no penalty for the option prepayment. DerivaGem tool can model only constant penalty over time. If penalty is $x\%$, then the strike price is set at $K = 100 \times (1 + x\%)$;
5. Option life (in years). Option life typically equals to the maturity term of the bond minus notice period of the option minus one day;
6. Short-rate volatility of the market yields. The market yield rates are used to estimate the short-rate discount factors used in bond valuation. The volatility is estimated based on historical behavior of the short-rate (3-months) yield series. The credit risk of the yield series is selected to match the credit rating of the reference bond to take into account the credit risk exposure in the option transaction (if the bond defaults, then the payouts in the options are set to zero). Alternatively, the volatility can be estimated using treasury rates (assuming that valuation is performed using treasury rates) but the default state of the bond and the varying probability of default must be modeled explicitly. For Hull-White (extended Vasicek) model volatility is estimated as $\hat{\sigma} = \sqrt{250} \times stdev\,[dr_t]$;
7. Mean-reversion rate. The default mean-reversion rate was assumed to be equal to zero;
8. Tree steps. Generally, is set to four times the maturity term. Factor four corresponds to a 3-month length of each discrete time period in the interest rate tree;
9. Term structure. Each model of the short-rate (Hull-White extended Vasicek or extended CIR) produces a term-structure specific to the model. For example, the term structure for the Hull-White extended Vasicek model, assuming zero mean-reversion rate, is equal to $R_T = r_0 + \frac{1}{2}\vartheta T - \frac{1}{6}\sigma^2 T^2$. The term structure must be set in the corresponding cells. To estimate the term structure, the drift parameter ϑ and the initial value r_0 must be set. The drift parameter ϑ effectively determines the slope of the term structure. The parameter is calibrated from the term structure estimated as of the valuation date: $\vartheta = \frac{2}{T}(R_T - r_0) + \frac{1}{3}\sigma^2 T$. Parameter r_0 is set to match the bond price to the quoted bond price as discussed below;
10. Quoted Bond Price. If the valuation is performed as of the bond issue date, the quoted bond price is generally set to par (100) value. The par value must be consistent with the term structure parameters of the model. We calibrate the parameter r_0 so that the quoted bond price equals to bond par value.

[39] In most cases our tool produced the option prices with less than 1 bps difference from the DerivaGem tool.

11. Frequency of the coupon payments is set to semi-annual to match the coupon frequency of standard USD bonds. Generally, the parameter must match the frequency of the bond coupon payments;
12. The interest rate model is set to “Normal – American”.

Application of the DerivaGem tool is illustrated in the diagram below.



E.2 Comparing DerivaGem tool to our ac.iOption tool

The list below summarizes the differences between our tool and DerivaGem tool.

- DerivaGem American-Normal model corresponds to the Hull-White (extended Vasicek) model. DerivaGem American-Lognormal model does not correspond to Hull-White (extended CIR) model and uses instead $\sigma(r_t) = \sigma \times r_t$ volatility function. The Lognormal model has the properties, which are similar to the Hull-White (extended CIR) model properties: (i) the set of interest rates is bounded by zero from below and (ii) the volatility of interest rates is heterogeneous and increases with the increase in interest rates;

- DerivaGem tool does not take into account various options and provisions such as (i) interest deferral or (ii) bond amortization schedule, which can affect materially the results of the option calculations;
- Only simple redemption penalty structure can be set in the DerivaGem tool. A standard penalty structure described in the Appendix D cannot be set within the DerivaGem tool;
- DerivaGem tool calculates only the option price but does not calculate the annuity adjustment factor. Therefore, the option price cannot be converted into the equivalent option premium / discount.
- Our tool is built as an extension of a generic optimization tool for one-dimensional Markov processes. It can be directly extended to other types of interest rate models or applied for other stochastic optimization models. DerivaGem tool can be applied exclusively to Normal and Lognormal interest rate models.

E.3 Estimation of Normal-American model in DerivaGem

Normal-American model in DerivaGem implements Hull-White (extended Vasicek) model, which is described by the following equation

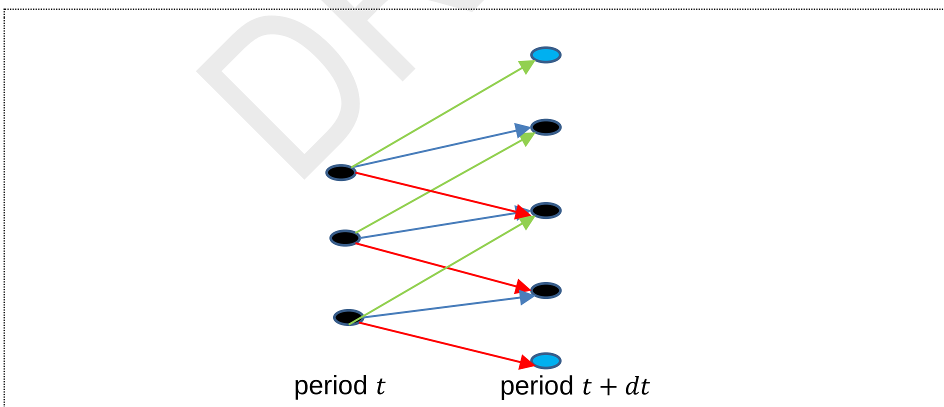
$$dr_t = (\theta_t - \alpha r_t)dt + \sigma dW_t$$

Mean-reversion parameter is assumed to be zero, $\alpha = 0$.

E.3.1 Set of states

The set of states in DerivaGem model is illustrated in the diagram below.

Exhibit E.1 Set of states in Hull-White (extended Vasicek) modelled by DerivaGem



The diagram shows the states in periods t and $t + dt$ The states in each period are distributed uniformly with the distance between each two states equal to

$$dr_{t,i} = \sigma \times \sqrt{3dt}$$

The states in period $t + dt$ are constructed by shifting all states in period t uniformly by $\vartheta_t dt$ and adding two additional states highlighted in the diagram with blue color. A uniform shift of all states by parameter $\vartheta_t dt$ ensures that the mean average state increases by $\vartheta_t dt$. A uniform distance $\sigma \times \sqrt{3dt}$ between the states ensures that no new states (with the exception of the two states highlighted in blue) are created in period $t + dt$. Note that this property holds only if the mean-reversion parameter is set to zero.

E.3.2 Transition probabilities

Suppose that the movement in the stochastic component of the process in each period is modelled using as follows:

$$dW_t \Rightarrow \begin{cases} \sqrt{3dt} & p_u \\ 0 & p_m \\ -\sqrt{3dt} & p_d \end{cases}$$

where p_u, p_m, and p_d ($p_u + p_m + p_d = 1$) are the probabilities of the stochastic component to move up, down, or stay the same. To be consistent with the standard Brownian motion, the p_u, p_m, and p_d probabilities must satisfy the following system of equations.

$$\begin{cases} (p_u - p_d) \times \sqrt{3dt} = 0 \\ (p_u + p_d) \times (3dt) = dt \end{cases}$$

where the two equations ensure that the mean and the variance of the standard Brownian motion match the mean and the variance of its discrete approximation with the trinomial tree. The solution to the system of equations is

$$\begin{cases} p_u = p_d = \dfrac{1}{6} \\ p_m = \dfrac{2}{3} \end{cases}$$

E.3.3 Summary

To summarize:

- Hull-White (extended Vasicek) model uses uniformly distributed set of states with the step $\sigma\sqrt{3dt}$;
- The set of states moves over time consistently with the drift function $\vartheta_t dt$;
- The transition probabilities $p_u = \frac{1}{6}, p_m = \frac{2}{3}$, and $p_d = \frac{1}{6}$ are set at constant values in each state so that the volatility of the numeric process is consistent with the volatility of the theoretical process.

E.4 Estimation of Lognormal-American model in DerivaGem

Normal-American model in DerivaGem implements Hull-White (extended Vasicek) model, which is described by the following equation

$$dr_t = (\theta_t - \alpha r_t)dt + \sigma r_t dW_t$$

Mean-reversion parameter is assumed to be zero, $\alpha = 0$.

E.4.1 Set of states

The set of states in each period t is log-uniform:

$$\frac{r_{t,i+1}}{r_{t,i}} = a$$

Equivalently the condition implies that the ratio of each two consecutive states is constant. The states in period $t + dt$ are constructed by multiplying each period t state by the same factor ρ_t so that the ratio of two consecutive states in period $t + dt$ equals to the ratio in period t. Parameter ρ_t is estimated so that to match the drift parameter of the interest rate process.

E.4.2 Transition probabilities

The change in the process between periods t and $t + dt$ is described by the following equation

$$r_{t+dt} \Rightarrow \begin{cases} a \times (1+\rho_t) r_t & p_u \\ (1+\rho_t) r_t & p_m \\ \dfrac{1}{a} \times (1+\rho_t) r_t & p_d \end{cases}$$

The average value of the interest rate process in period $t + dt$ equals

$$\mu_{t+dt} = \mu_t + \vartheta_t dt = (1+\rho_t) \times \mu_t$$

Therefore

$$\rho_t = \frac{\vartheta_t dt}{\mu_t}$$

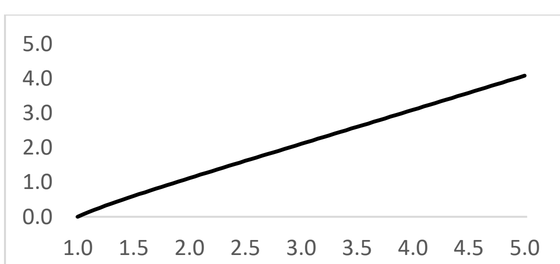
DerivaGem uses the same symmetric transition probabilities $p_u = \frac{1}{6}, p_m = \frac{2}{3}$, and $p_d = \frac{1}{6}$ in both Normal and Lognormal models. For the Lognormal model we estimate parameter a which is consistent with the transition probabilities and respective volatility of the random variable r_{t+dt}. Volatility of the random variable r_{t+dt} is described by the following equation

$$\sigma(r_{t+dt}) = (1+\rho_t) r_t \times \sqrt{\frac{1}{6} \times (a-1)^2 + \frac{1}{6} \times \left(\frac{1}{a} - 1\right)^2} = r_{t+dt} \times \frac{a-1}{a} \sqrt{\frac{1}{6} \times (1+a^2)} = r_{t+dt} \times \sigma \times \sqrt{dt}$$

Parameter a is estimated implicitly from the equation as follows:

$$\frac{a-1}{a} \times \sqrt{\frac{1+a^2}{2}} = \sigma \times \sqrt{3dt}$$

Note that function $f(a) = \frac{a-1}{a} \times \sqrt{(1+a^2)}$ is an increasing function of a for $a > 1$ and therefore the above equation has a single solution. The function $f(a)$ is presented in the diagram below on the $a \in [1,5]$ interval.



Alternatively, for small values of dt, the value of a can be approximated as follows:

$$a = 1 + \sigma \times \sqrt{3dt}$$

E.4.3 Summary

To summarize:

- Lognormal model uses log-uniformly distributed set of states with the step $\frac{r_{t,i+1}}{r_{t,i}} = a \sim 1 + \sigma \times \sqrt{3dt}$;
- The states are multiplied by a constant factor $1 + \rho_t = 1 + \frac{\vartheta_t dt}{\mu_t}$ over time consistently with the drift function $\vartheta_t dt$ and to ensure that the log-uniform distribution of states is preserved;
- The transition probabilities $p_u = \frac{1}{6}, p_m = \frac{2}{3}$, and $p_d = \frac{1}{6}$ are set at constant values in each state so that the volatility of the numeric process is consistent with the volatility of the theoretical process.

Appendix F ac.finance.SRM Tool

In this section we provide a short description of the ac.finance.SRM tool that was developed as part of this guide and can be downloaded from the alexacomputing.com website. The modelling part of the interest rate option is implemented in java and the interface is implemented in Excel through custom functions that are executed by java virtual machine. A detailed description of the tool architecture can be found in alexacomputing.com.

F.1 Overview

The tool is developed based on the modelling framework for controlled one-dimensional Markov processes. A controlled Markov process generation requires a specification of the following parameters: set of states, transition probabilities, discount rates, objective function and other. The parameters are specified in the ac.SRM tool as follows.

1. Transition probabilities are modelled based on the diffusion process specification of the interest rate process. Transition probabilities are modelled either using the functional form (derived from the diffusion process specification) or using the matrix form (if the diffusion process is modelled as a discrete tree). Default option is to model the transitional probabilities using respective functional form which describes the normal distribution of transition probabilities.
2. Set of states is modelled either as a tree or a discrete collection of points in a one-dimensional interval [a, b] (which is the default option). The interval [a, b] and the collection of points are selected consistently with the transition probabilities to ensure that the probability that the interest rate process will stay within the interval is close to one.
3. Set of modes is modelled as a binary set {0, 1}, where mode zero corresponds to non-exercised and mode 1 corresponds to exercised option.
4. Discount rates and discount factors are calculated based on the modelled interest rate process.
5. Objective function is modelled based on the cash flows of the underlying bond instrument.

Currently the tool is limited to modelling four specific processes (but can easily be extended for other interest rate model specifications):

(i) Two parametric models (Vasicek and CIR); and

(ii) Two arbitrage-free models (Hull-White extended Vasicek and Hull-White extended CIR).

The tool implements the above models by modelling the parameters based on the selected model. The tool also implements model validation, which includes validation of (i) process distribution and (ii) bond bullet prices.[40]

Modelling is implemented using the following steps:

1. Estimation of the selected interest rate process parameters.
2. Construction of the interest rate tree.
3. Construction of the interest rate calculator; and

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[40] The implemented models have closed-form equations for the process distribution and bullet prices, which are compared against the numerically calculated values. The closed-form equations are summarized in Section 2.2 and Section 3.

4. Modelling and pricing interest rate derivatives.

The parameters estimation and option valuation steps are implemented and tested independently by separate java calculator classes. Parameters are estimated using as inputs (i) the historical 3-month yield series (to estimate the process volatility parameter); and (ii) the yield term structure (to estimate the drift parameter). The mean-reversion parameter is set by default to zero.

F.2 Parameter estimation

Parameter estimation is a key step to produce reasonable option values. A historical change in the yield rates (and respective model parameters estimated under different methods) is presented in Appendix H. This section described how to select the tool inputs to estimate model parameters under different approaches.

Conceptually, there are two alternative parameter estimation methods:

1. **Random walk model**. Under the approach, mean-reversion parameter is assumed to be zero, volatility is estimated based on historical 'change in yield' sample, and drift parameter is estimated based on the yield term structure.
2. **Mean-reversion model**. Under the approach, mean reversion and drift parameters are estimated based on historical sample and assumptions on the long-term equilibrium and the volatility parameter is estimated based on the historical sample of residuals. Mean –reversion is modelled using HP filter modelling.

F.2.1 Sample parameters

As a first step, a sample is selected, which is used for the model parameters estimation. Normally, a short-term (3-months) yield series is selected with the industry sector matching the borrowing entity industry sector (e.g. Industrial, Financial, or other) and the credit rating of the series matching the tested transaction credit rating.

After a series is selected for parameters estimation, the following sample parameters must be specified.

1. Sample size parameter n. The larger the sample size, the more significant is the dependence of the parameters on the market long-term behavior and less dependence on recent changes in the market. The parameter is set using the 'sample-size' keyword.
2. Probability threshold to remove the sample outliers. The changes in the yield series are assumed to have a normal distribution. If a certain change is too large and not consistent a change in a normal distribution, the observation is removed from the sample. Outliers are normally identified for a yield series with daily frequency. The parameter is set using the 'outlier-prob-threshold' keyword.
3. Parameter τ, which is the period between consecutive yield series observations applied to estimate the change in the yields series. The parameter is set using the 'volatility-sample-period' keyword. The larger is the τ parameter, the less is the yield series affected by the outliers.

F.2.2 Random walk model

The approach is applied if the value of the 'state-long-term' is empty (or zero). The parameters are estimated as follows under the approach.

(i) **Mean-reversion**. Mean-reversion parameter is automatically set to zero.

(ii) **Drift**. If term structure parameter is empty, drift parameter is set at zero. Otherwise, drift parameter is estimated using equations described in Section 3.4. Parameter T in the equations is set using the 'term-structure-max-tenor' parameter key. By default, the parameter is set equal to the tested loan maturity term. If options are estimated for multiple scenarios, the parameter should be set consistently across the scenarios (e.g. set to maximum maturity term in the scenarios).

(iii) **Volatility**. Volatility may be estimated using multiple approaches as discussed in Section 3.3. The volatility estimation method is set using 'volatility-method' key, which values can be set to the following list: 'volatility-method-stdev', 'volatility-method-ewma', or volatility-method-garch'. The default value is 'volatility-method-stdev', which corresponds to volatility estimation approach based on a standard deviation of the historical yield change series. The volatility estimates are described by equations (3.3) and (3.4).

The parameters estimates can be overridden manually.

F.2.3 Mean-reversion model

The approach is applied if the value of the 'state-long-term' is non-empty (or non-zero). The parameters are estimated as follows under the approach.

(i) **Mean-reversion**.

(ii) **Drift**.

(iii) **Volatility**. Volatility parameter is estimated based on the HP residuals using the methods described in the previous section.

F.3 Single option

The option modelling is implemented either (i) as a collection of two functions (where the first function models option parameters and the second function models the option value); or (ii) as a tool, which implements detailed steps of option modelling.

The option estimation is performed by executing the following steps:

1. Select the interest rate model and estimate the model parameters;
2. Set the option parameters and estimate the option;
3. Review the model validation output and review the results of the option estimation.

Implementation of the above steps is discussed below.

F.3.1 Option implementation via functions

The implementation via functions was included so that option calculation could be added directly to other tools. The steps 1 and 2 are implemented respectively as two functions discussed below. Validation of the function implementation was performed by comparing the functions output against the tool output.

Function for model parameters estimation

Function for option value calculation

F.3.2 Option implementation as a tool

Below we show the tool print screens to illustrate the application of the tool. The example is illustrated for the Hull-White (extended Vasicek) model. The tabs highlighted in yellow are the input tabs.

1. **Input tabs**.

   - Input tabs ('ylds' and 'sample') are highlighted in yellow. The 'ylds' tab includes the data on the yield term structure and historical 3-month yield sample.
   - The key model inputs are set in the 'Parameters' and 'Summary' tabs. Auxiliary input parameters are set in the 'calc' tab. Specifically,
   - The objective of the 'Parameters' tab is to estimate the underlying stochastic model parameters such as process volatility and drift parameters;
   - The parameters, which are specified in the 'Summary' tab, describe the terms and conditions of the tested transaction.
   - Parameters set in other tabs specify the implementation of the interest rate tree and specify whether certain parameters of the model are calibrated to ensure the par value of the bullet bond.

   A more detailed description of the model parameters is provided below.

2. **Parameters "params" tab**.

   The process parameters on the tab are estimated as follows:

   - The volatility parameter is estimated based on the sample of short-term (3-months) yield rates with the credit rating matching the credit rating of the tested transaction. The sample size of the 3-month yield sample is specified using the sample-size keyword. The estimated volatility parameter can be manually overridden by the user by specifying the custom value in the yellow highlighted cell. By default, the sample size parameter is set to 250 (approximate number of business days in a fiscal year). The tab includes a graph with volatility estimated for different sample sizes. The graph shows how materially the volatility depends on the sample size;
   - The drift parameter is estimated using the yield term-structure and the equations described in Section 3.4. For Vasicek and CIR models the estimated constant drift parameter can be overridden manually by the user. The estimated parameter depends on the parameter T which is set using the term-structure-max-tenor parameter. By default, the parameter is set equal to the tested transaction maturity term.

     For the Hull-White (Vasicek) and Hull-White (CIR) models the estimated drift parameter is a function and therefore cannot be overridden.
   - Mean-reversion parameter is by default set to zero.

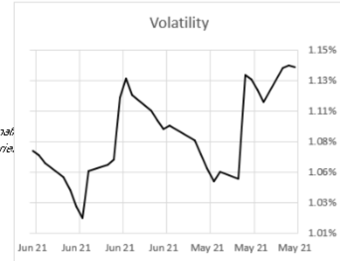
   The tab with estimated model parameters is shown in the diagram below.

Exhibit F.1 ac.iOption tab with parameters estimation output

TAB: OPTION PARAMETERS ESTIMATION

Date	Yield series	Date	Volatility series
23-Jun-21	3.44%	23-Jun-21	1.07%
22-Jun-21	3.33%	22-Jun-21	1.07%
21-Jun-21	3.24%	21-Jun-21	1.07%
18-Jun-21	3.44%	18-Jun-21	1.05%
17-Jun-21	3.39%	17-Jun-21	1.04%
16-Jun-21	3.42%	16-Jun-21	1.03%
15-Jun-21	3.48%	15-Jun-21	1.02%
14-Jun-21	3.39%	14-Jun-21	1.06%
11-Jun-21	3.32%	11-Jun-21	1.06%
10-Jun-21	3.40%	10-Jun-21	1.07%
9-Jun-21	3.51%	9-Jun-21	1.12%
8-Jun-21	3.56%	8-Jun-21	1.13%
7-Jun-21	3.48%	7-Jun-21	1.12%
4-Jun-21	3.35%	4-Jun-21	1.11%
3-Jun-21	3.43%	3-Jun-21	1.10%
2-Jun-21	3.27%	2-Jun-21	1.09%
1-Jun-21	3.42%	1-Jun-21	1.10%
28-May-21	3.43%	28-May-21	1.08%
27-May-21	3.76%	27-May-21	1.07%
26-May-21	3.92%	26-May-21	1.06%
25-May-21	3.87%	25-May-21	1.05%
24-May-21	3.72%	24-May-21	1.06%
21-May-21	3.94%	21-May-21	1.05%
20-May-21	3.83%	20-May-21	1.14%
19-May-21	3.92%	19-May-21	1.13%
18-May-21	3.63%	18-May-21	1.12%
17-May-21	4.07%	17-May-21	1.11%
14-May-21	3.68%	14-May-21	1.14%
13-May-21	4.12%	13-May-21	1.14%
12-May-21	4.11%	12-May-21	1.14%
11-May-21	3.82%		
10-May-21	4.00%		
7-May-21	4.03%		
6-May-21	4.03%		
5-May-21	3.66%		
4-May-21	3.68%		
3-May-21	3.24%		
30-Apr-21	3.29%		
29-Apr-21	3.35%		
28-Apr-21	3.24%		
27-Apr-21	3.05%		
26-Apr-21	3.16%		
23-Apr-21	3.16%		
22-Apr-21	2.79%		
21-Apr-21	3.30%		
20-Apr-21	3.41%		
19-Apr-21	3.42%		
16-Apr-21	3.27%		
15-Apr-21	2.95%		
14-Apr-21	2.96%		
13-Apr-21	3.84%		
12-Apr-21	3.61%		
9-Apr-21	2.84%		
8-Apr-21	4.12%		
7-Apr-21	4.13%		
6-Apr-21	3.93%		

Parameter	Value	Description
Option model	Vasicek	
Parameter estimation model	random-walk	
Historical sample		
sample-size	125	*Sample size in number of business days (for σ calculation)*
volatility-sample-period	32	*Period between consecutive yields (for change in yields estimat*
outlier-prob-threshold	0.010%	*The probability thershold applied to identify 3-month yield serie*
sample-size-min	30	*Minimum sample size used for volatility estimation*
Volatility		
volatility-method	volatility-method-stdev	
Drift		
term-structure-max-tenor	3.17	*Tenor for θ calibration (default: set to maturity)*
Mean-reversion		
state-long-term	4.80%	*Long-term yield equilibrium (for mean-reversion estimation)*

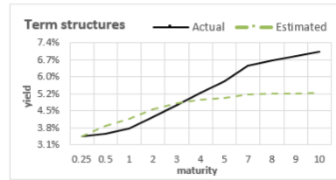


Parameter	Estimated	Override	Final	Notation	LT equilibrium	Value
volatility	1.41%	1.41%	1.41%	σ	Average yield	3.53%
drift	5.45%	5.45%	5.45%	ϑ	Current yield	3.44%
mean-reversion	99.00%	99.00%	99.00%	α		
drift-function	java.util.LinkedHashMap@Sun, 27 Jun 2			12	*Final (post-overriding)*	

Yield term-structure (Bloomberg, *Estimated yield term structure (using final parameters)*

srm-functions-estimate java.util.ArrayList@Sun, 27 12 3

Actual			Estimated				
Index	Tenor	Yield	Period	Yield	Drift	Drift Cum	Difference
1	0.25	3.44%	0.25	3.44%	5.45%	5.45%	0.00%
2	0.5	3.57%	0.50	3.88%	5.45%	6.81%	-0.31%
3	1	3.76%	1.00	4.19%	5.45%	9.53%	-0.43%
4	2	4.26%	2.00	4.60%	5.45%	14.98%	-0.35%
5	3	4.78%	3.00	4.84%	5.45%	20.42%	-0.06%
6	4	5.30%	4.00	4.98%	5.45%	25.87%	0.32%
7	5	5.77%	5.00	5.08%	5.45%	31.31%	0.69%
8	7	6.43%	7.00	5.20%	5.45%	42.20%	1.23%
9	8	6.66%	8.00	5.23%	5.45%	47.65%	1.42%
10	9	6.84%	9.00	5.26%	5.45%	53.09%	1.58%
11	10	7.04%	10.00	5.28%	5.45%	58.54%	1.75%
12	15	7.87%	15.00	5.35%	5.45%	85.77%	2.52%
13	20						
14	25						
15	30						



Average drift parameter		
	5Y	6.26%
	10Y	5.85%
	20Y	

A. Parameter calculator parameters

Historical sample	125	java.util.ArrayList@Sun, 27 Jun 2021 09:10:45:243..7075598971	123
Term structure		java.util.LinkedHashMap@Sun, 27 Jun 2021 09:10:45:247..146(	15
Calculator parameters		java.util.LinkedHashMap@Sun, 27 Jun 2021 09:13:12:176..7874	11

B. Parameter calculator

SRM parameter calculator		jeconkr.finance.HW.lib.ch23_srm.calculator.parameter.CalculatorParameterVasicek@Sun, 27 Jun 2021 09:14:26:506..1641504762454		
SRM parameters	srm-params	java.util.LinkedHashMap@Sun, 27 Jun 2021 09:13:12:185..6576	6	
volatility-function		java.util.ArrayList@Sun, 27 Jun 2021 09:10:45:324..5842263954	62	*Estimated (prior to overriding)*
number of eliminated observations (in volatility estimation)			34	

C. Override manually model parameters

SRM parameter calculator		jeconkr.finance.HW.lib.ch23_srm.calculator.parameter.CalculatorParameterVasicek@Sun, 27 Jun 2021 09:14:26:510..4460860874138	
SRM parameters	srm-params	java.util.LinkedHashMap@Sun, 27 Jun 2021 09:13:12:189..8768	6

3. **Summary ("summary") tab**. The tab with the option parameters and option estimation output is shown in the diagram below.

   The tab includes both the input parameters and the estimation output. The following input parameters are used:

   - Terms of the testes transactions such as valuation date, credit rating, maturity term, length of the tree step (*dt*), and coupon payment parameters (fixed coupon rate and coupon payment frequency).

     The valuation date and credit rating are specified for reference purpose only. The yield term structure and 3-month yield sample (used for volatility and drift parameter estimation) must be consistent with the valuation date and credit rating parameters.

     Default length of the interest rate tree is set at *dt=0.25* to match a 3-month period.

     By default, the coupon rate is replaced with the estimated coupon rate which corresponds to the par bullet value of the bond.

Semi-annual coupon frequency is used as a default value to match the frequency of the US$ bond yield rates (SA frequency, 30/360 day count).

- ► Bond redemption premium schedule.
- ► Other terms which effect the bond cash flows such as bond amortization schedule or bond interest payment deferral.

The following output is presented in the tab:

- ► Call/put option fixed price, annuity adjustment factor, and call/put option annual premium. The annual premium is used to adjust the yields on callable bonds;
- ► The price of the callable/putable and bullet bonds. The call/put price is calculated as the difference between the prices of the callable/putable and the bullet bonds;
- ► Results of the estimation validation procedure which include: comparing the numerical and theoretical (i) bond bullet prices; (ii) terminal state distribution mean; and (iii) terminal state distribution standard deviation. In addition, the validation procedure estimates the implied drift/volatility parameters based on the numerical terminal state distribution and compares the implied values to the values used in the model. The purpose of the validation procedure is to test that (i) the numeric tree was constructed correctly and to test that (ii) the backward recursion in the calculator (applied to calculate bond prices) is implemented correctly.

The tool tab is presented below.

Exhibit F.2 ac.iOption tab with the output of the option estimation

TAB: SUMMARY OF THE RESULTS

Model Parameters

Parameter	Value	Description
Interest rate model	Hull-White, extended Vasicek	
Valuation date	1-Jan-19	
credit rating	BB-	Bond transaction-specific credit rating
maturity term	5.0	Bond maturity term
coupon-rate	4.40%	Calibrated coupon rate
coupon-frequency	2	Coupon payment frequency
dt	0.25	Length of the tree step (in years)
drift parameter (ϑ)	N/A	Calibrated from the term structure
mean-reversion parameter (α)	0.000%	Default value is set to zero
volatility (σ)	0.759%	Estimated using historical data sample
principal-amount	100	Bond principal amount
number of periods	20	Number of tree steps

Model Output Summary

Parameter	Value
Bond actual value	-99.10
Bond bullet value	-100.00
Bond option value	**0.902**
Annuity adjustment factor (AAF)	2.774
Bond option premium / discount	**0.325**

Model Validation Summary

	Zero-price bond	*Terminal distribution parameters*		*Implied model parameters*	
Model type	Bullet price	Terminal mean	Terminal stdev	Implied mean	Implied stdev
Numeric model	100.00	6.47%	1.69%	0.68%	0.76%
Theoretic model	99.70	6.47%	1.70%	0.68%	0.76%
Difference	***0.30***	***0.00%***	***-0.01%***	***0.00%***	***0.00%***

Markov process models

Vasicek
Cox-Ingersoll-Ross (CIR)
Hull-White, extended Vasicek
Hull-White, extended CIR

Interest rate options

call
put

BOND SCHEDULES

Bond option parameters

Parameter	Value	Description
option	call	Option type (put or call)

Bond redemption premium / discount

Year	Period	Value	Description
0.25	1	0.00%	Bond redemption premium in period t=1
2.5	10	0.00%	Bond redemption premium in period t=10
5.0	20	0.00%	Bond redemption premium at maturity

redemption-premium-schedule
java.util.LinkedHashMap@Tue, 5 Fe Redemption premium if option is exercised

Bond amortization schedule

Year	Period	Value	Description
0.25	1	0%	% of bond amortization in period t=1
1.75	7	0%	% of bond amortization in period t=7
3.5	14	0%	% of bond amortization in period t=14
5.0	20	100%	% of bond amortization at maturity

amortization-schedule
java.util.LinkedHashMap@Tue, 5 Fe Bond amortization schedule

Interest deferral schedule

Parameter	Value	Description
interest deferral period	0	The latest coupon payments deferral period
is compounded	1	1 if deferred interest is compounded (0 otherwise)

Controls

Ctrl+Shft+C	Ctrl+Shft+D	Ctrl+Shft+S	Ctrl+Shft+V
CALCULATE	DISABLE	SET RANGES	SAVE AS VALUES

4. **Calculator tab**

The calculator tab specifies the java functions which are used to construct the option tree model and the option tree calculator. The tab also includes some auxiliary but important parameters. The parameters are set to default values by the tool by can be modified by the user on the tab. The list of auxiliary inputs includes the following parameters:

- ► Tree state parameters specified by *dx-down, dx-up, X-min, X-max* and *X-count* keys. The parameters specify (i) the distance between the neighbour states; (ii) the minimum and maximum states; and (iii) the number of states used to model the tree;
- ► The ***adjust-to-par*** parameter specifies whether the bond fixed coupon rate specified by the user is replaced by the model-estimated rate to ensure that the bullet bond is values at par. In a typical case, the option is evaluated at the bond issue date and therefore the bullet bond must be priced at par. If however the option is valued at a date different from the issue date, then the parameter should generally be set to zero.
- ► The initial state of the tree model is set equal to the 3-month yield rate of the term structure.
- ► The tab also includes the parameter which specifies whether a tree or a grid structure is sued to construct the tree. The grid structure assumes a uniform set of states in each period t.

5. **Output tabs**. In addition to the 'summary' tab the output tabs include(i) 'cf' tab with the bond cash flows; (ii) 'charts' tab (presented below); (iii) 'value' tab with the tree of the callable/putable bond values; (iv) 'bullet' tab with the tree of the bullet bond values; (v) 'action' tab with the option exercise / not exercise action tree; and (vi) 'D' tab with the option tree state distribution probabilities.

   The output of the 'charts' tab is presented in the exhibit below. The tab shows two graphs (i) the terminal state distribution; and (ii) the state tree including state bounds, fixed (calibrated) bond coupon rate; average yield rate; and the boundary between the option exercise / not exercise actions.

   The exercise / not exercise boundary increases over time and converges to the bond coupon rate.

Exhibit F.3 ac.iOption tab with the output charts

TAB: CHARTS

State distribution at maturity

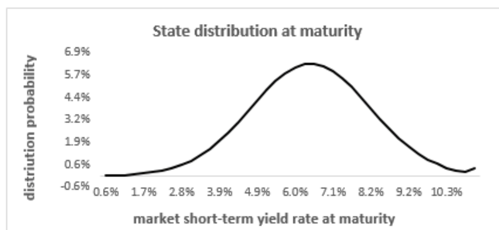
number of periods	21
number of observations	40

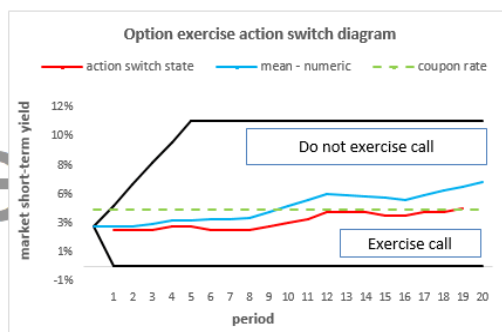
Option exercise action switch state

Mean:	6.47%	dt	0.25
StDev:	1.69%		

#	state	prob	period (years)	period (index)	action switch state	mean - numeric	coupon rate	min state	max state
1	0.6%	0.01%				3.07%	4.40%	3.07%	3.07%
2	0.9%	0.03%	0.25	1	2.80%	3.07%	4.40%	0.00%	4.68%
3	1.2%	0.05%	0.50	2	2.80%	3.07%	4.40%	0.00%	6.29%
4	1.5%	0.08%	0.75	3	2.80%	3.30%	4.40%	0.00%	7.90%
5	1.7%	0.13%	1.00	4	3.07%	3.53%	4.40%	0.00%	9.51%
6	2.0%	0.20%	1.25	5	3.07%	3.57%	4.40%	0.00%	11.12%
7	2.3%	0.30%	1.50	6	2.80%	3.62%	4.40%	0.00%	11.12%
8	2.5%	0.43%	1.75	7	2.80%	3.66%	4.40%	0.00%	11.12%
9	2.8%	0.61%	2.00	8	2.80%	3.71%	4.40%	0.00%	11.12%
10	3.1%	0.85%	2.25	9	3.07%	4.17%	4.40%	0.00%	11.12%
11	3.3%	1.15%	2.50	10	3.33%	4.64%	4.40%	0.00%	11.12%
12	3.6%	1.52%	2.75	11	3.60%	5.10%	4.40%	0.00%	11.12%
13	3.9%	1.96%	3.00	12	4.14%	5.57%	4.40%	0.00%	11.12%
14	4.1%	2.46%	3.25	13	4.14%	5.46%	4.40%	0.00%	11.12%
15	4.4%	3.02%	3.50	14	4.14%	5.35%	4.40%	0.00%	11.12%
16	4.7%	3.62%	3.75	15	3.87%	5.25%	4.40%	0.00%	11.12%
17	4.9%	4.22%	4.00	16	3.87%	5.14%	4.40%	0.00%	11.12%
18	5.2%	4.80%	4.25	17	4.14%	5.47%	4.40%	0.00%	11.12%
19	5.5%	5.33%	4.50	18	4.14%	5.80%	4.40%	0.00%	11.12%
20	5.8%	5.77%	4.75	19	4.41%	6.14%	4.40%	0.00%	11.12%
21	6.0%	6.10%	5.00	20		6.47%	4.40%	0.00%	11.12%
22	6.3%	6.28%							
23	6.6%	6.31%							
24	6.8%	6.18%							

Charts





6. **Other tabs**. Other tabs include the 'config' and 'validate' tabs. The 'config' tab includes a full list of input and output parameter keys. The 'validation' tab presents more detailed results of the model validation analysis.

F.4 Sample of options

In a typical IRB analysis, not only the tested transaction is a callable transaction but many CUTs, included in the sample to estimate the arm's length interest rate, are also callable. Therefore, the yield rates on the CUTs must also be adjusted.

A natural approach to calculate a sample of options is to create an option calculator for each individual option. Note however that option estimation is in general both time and memory intensive task. Therefore, calculation of a sequence of options may be time and memory intensive process.

The sample option calculator tool, included as part of ac.iOption package, makes simplifying assumptions on the CUTs prepayment options. The assumptions ensure that the option tree constructed for the tested transaction can also be applied for the sample CUTs.[41] The following simplifying assumptions are made:

1. Maturity term. Remaining maturity term for each CUT (denoted as T_i) is set equal to the maturity term of the tested transaction (denoted as T);
2. Volatility and drift parameters. The same parameters are used as the volatility and drift parameters applied for the tested transaction. (Note that parameters are estimated based on the credit rating of the tested transaction);
3. Coupon rate. The coupon rate is calibrated to ensure that the bullet value of the bond equals to the par value;
4. Penalty structure. The penalty structure of each CUT is modified as follows:
   - The remaining duration of the make-whole provision is rescaled to match the modified maturity term of the CUT. If T_i^{mw} denotes the remaining duration of the make-whole provision, then the modified duration is calculated as follows: $\tilde{T}_i^{mw} = T_i^{mw} \times \frac{T}{T_i}$;
   - The post-make-whole penalty structure is typically a linear function. The redemption penalty is reduced from x_i (which is applicable at the make-whole termination date) to zero (which is applicable at the maturity date). The value x_i is modified to ensure that the slope of the penalty structure prior to and after adjustments are the same. The modified penalty $\tilde{x}_i$ is calculated as follows: $\tilde{x}_i = x_i \times \frac{T - \tilde{T}_i^{mw}}{T_i - T_i^{mw}}$.

Effectively the assumptions imply that a callable tested transaction and CUT are comparable to each other in terms such as maturity, coupon rate, and interest rate process parameters and the difference in the tested transaction and CUT options is due only to differences in the penalty structure.[42]

The sample option calculator recalculates the bond cash flows for each individual CUT based on the CUT prepayment penalty structure and then recalculates the CUT option value and respective option annual premium. The yield on the callable CUT is adjusted downward by the option premium value.

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[41] Construction of the option tree is the most time and memory intensive part of the option valuation process.

[42] If the credit ratings and tenor terms of the CUTs are selected so that to match the terms of the tested transaction, then the assumptions do not change significantly the CUTs call option adjustment. However, the assumption that the CUTs are priced at par on the valuation date of the tested transaction is generally not true. The exact estimation of the CUTs prepayment option may result in the elimination of some of the CUTs from the sample due to the fact that the CUTs should have been prepaid according to the option valuation model. Note that the spread between the yield and coupon rate is one of the criteria which is applied in sample screening.

The objective of the approach is to avoid an in-depth assessment of each individual CUT prepayment provisions. If both the CUT and the tested transactions are callable, then the yield on the CUT is either not adjusted or adjusted (partially or fully) only if the penalty structure of the CUT prepayment option makes it expensive to exercise the option.

F.5 Pre-calculated options

An alternative approach to estimating a sample of the options is to create a single table (database) of options pre-calculated for a range of different option parameters. Each specific option is calculated then by (i) matching the option parameters to the parameters included in pre-calculated option database; and (ii) interpolating the matched option prices. The following parameters determine the option value:[43]

1. Volatility parameter;
2. Drift parameter;
3. Maturity term;
4. Make-whole-termination date;
5. Post-make-whole redemption penalty;
6. Difference between the coupon and the yield rates;

In total there are seven parameters that determine the value of the option. Suppose that five different values are assumed for each parameter. Then the total number of different pre-calculated options is

$$n = 5^7 = 78{,}125$$

For each specific set of parameters, $k = 2^7 = 128$ neighbor parameter sets must be identified, and the option value is estimated by interpolating the 128 pre-calculated option values for the neighbor sets of parameters.

[43] The list includes only the parameters which have the most material impact on the option value.

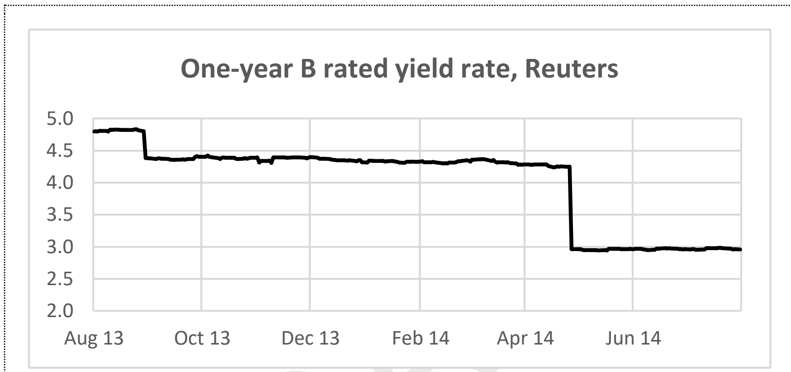
Appendix G Examples

xx

G.1 Volatility estimation

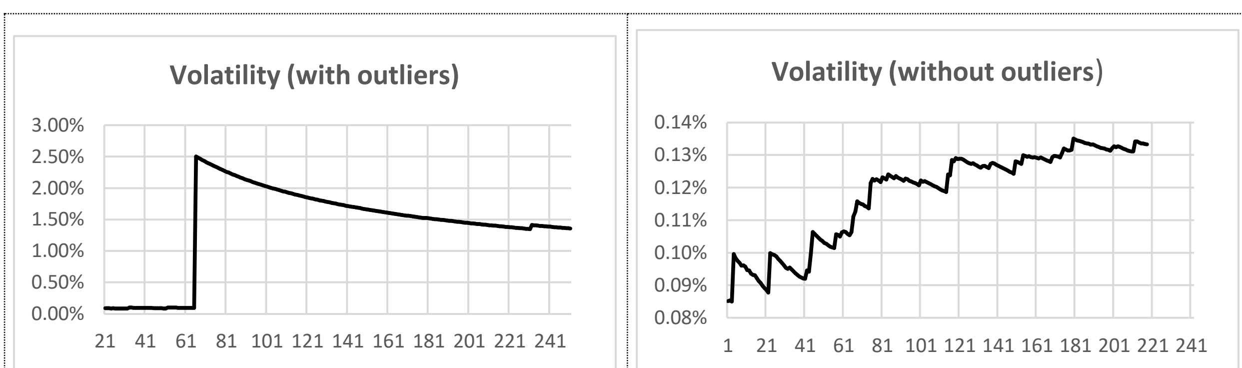
Volatility estimate based on historical short-term yield data can potentially be very sensitive to the yield outliers. To illustrate it by an example, we provide below an example in which the volatility is estimated based on August 2013 – August 2014 historical B rated one-year[44] yield series as reported by Reuters. The yield series is shown in the exhibit below.

Exhibit G.1 Short-term yield series used in volatility estimation



The series shows a large downward jump in the yield rates from 4.25% on 7 May 2014 to 2.96% on 8 May 2014. The jump is due to the change in the Reuters yield estimation methodology and does not represent actual volatility in the yield rates. Reuters adjusted materially all sub-investment yield series on that date. The exhibit below shows how the downward adjustment affects the estimated volatility parameter. The volatility series estimated for different sample sizes is presented in the exhibit below. The left panel shows the volatility prior to removing the yield outlier and the right panel shows the volatility after removing the outlier.

Exhibit G.2 Volatility prior to (left panel) and after (right panel) removing the outlier



The volatility graph on the left panel shows that there is an outlier that have a material impact on the estimated volatility. The estimated volatility parameter based on a sample of 250 business days was 1.36%.

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[44] Reuters did not report in 2014 the yield series with maturities shorter than one-year.

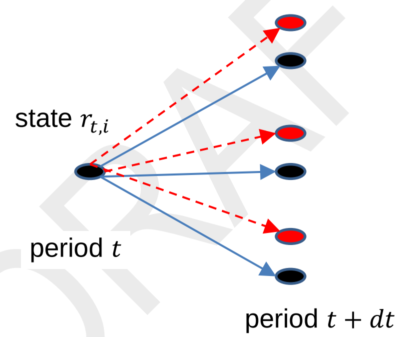
The volatility graph on the left panel shows the volatility estimate after removing the yield outliers. The volatility was reduced from 1.36% to 0.14% after four outliers were removed from the sample.[45]

The filter is implemented as follows:

1. Estimate the mean and standard deviation of the sample used in volatility estimation. Estimate the critical value such that the probability of a sample absolute value to exceed the critical value is smaller than some fixed alpha value (by default set at $\alpha = 10^{-5}$);
2. Remove all elements from the sample which absolute values exceed the critical value.
3. If outliers were identified, repeat step 1 until no new outliers identified or maximum number of iterations is exceeded (current maximum number of iterations is set at 10).

G.2 Transition probability estimation

The problem with the direct estimation of the transition probabilities is illustrated in the diagram below. The problem is illustrated with a trinomial tree example



The red lines in the diagram show the theoretical movement in the interest rate with $\mu = \vartheta_t dt$ upward drift. The black circles show actual discrete states used in the numerical modelling. At step one, prior to the transition probability adjustment, the red states are matched to the closest black sets and the probabilities are calculated based on the theoretical μ and σ values estimated at state $r_{t,i}$. After matching the red states to the actual black states, the drift $\mu = \vartheta_t dt$ in the interest rates is effectively reduced to zero. The impact on the volatility can also be generally material. Therefore, to ensure consistency with the theoretical model, the probabilities in the black states must be adjusted. Under the contraction mapping transition probability adjustment approach, the theoretical mean and standard deviation parameters μ and σ are replaced with the adjusted values $\tilde{\mu}$ and $\tilde{\sigma}$ such that the mean and standard deviation estimated based on the actual (black) discrete states match the theoretical μ and σ parameters.

The results of the transition probability adjustment are illustrated for the two alternative approaches discussed in Appendix B.3. The example was estimated using the Hull-White (extended Vasicek) model

[45] In practice volatility estimated based on Bloomberg series typically exhibits a much more regular behavior compared to the volatility estimated based on Reuters yield series. This is due to the fact that Reuters estimated the yields as Treasury rates plus risk spread, where the risk spread could be constant for an extended period of time and periodically adjusted materially up or down. As a result, the volatility estimated based on Reuters' yields can often be measuring the volatility of Treasury rates and be very low. (Since the last time we used Reuters' yields in the option valuation analysis, Reuters updated their methodology for yield series estimation. We have not tested the impact of the change in the methodology on the results of volatility estimation).

with the following parameters: $\alpha = 0, \sigma = 0.76\%$. The drift parameter was estimated based on the increasing term structure of yield rates with the equivalent constant drift parameter equal to $\vartheta = 0.68\%$.

Theoretical and numerical model parameters for unadjusted transition probabilities are described in the exhibit below.

Model name	μ_T	σ_T	μ	σ
This numerical model	3.19%	1.52%	0.03%	0.68%
Hull-White, extended Vasicek	6.47%	1.70%	0.68%	0.76%
Difference	***-3.28%***	***-0.18%***	***-0.66%***	***-0.08%***

The example illustrates that if the transition probabilities are unadjusted then the impact on both mean and standard deviation can be material (typically the impact on the mean is larger than the impact on standard deviation). In the example the drift parameter was reduced from $\vartheta = 0.68\%$ to a value close to zero ($\vartheta = 0.03\%$) consistently with the above diagram. The call option premium was estimated respectively at $call\ premium = 0.57\%$.

Next we adjusted transition probabilities using the “contraction mapping” approach. The numeric and theoretical parameters are described in the exhibit below.

Model name	μ_T	σ_T	μ	σ
This numerical model	6.47%	1.69%	0.68%	0.76%
Hull-White, extended Vasicek	6.47%	1.70%	0.68%	0.76%
Difference	***0.00%***	***-0.01%***	***0.00%***	***0.00%***

After adjusting the transition probabilities, the state distribution of the numerical model matches closely the theoretical state distribution. With the positive drift parameter, the call option premium reduced to $call\ premium = 0.33\%$. Due to precision, efficiency and robust performance, the transition probability adjustment based on contraction mapping estimation is applied as the default approach in the interest option valuation tool.

For completeness of the example exposition, we demonstrate also the performance of the transition probability adjustment algorithm based on solving the probability deviation minimization problem (described in the Appendix B.3). The numeric and theoretical parameters are described in the exhibit below.

Model name	μ_T	σ_T	μ	σ
This numerical model	6.32%	1.71%	0.65%	0.76%
Hull-White, extended Vasicek	6.47%	1.70%	0.68%	0.76%
Difference	***-0.15%***	***0.01%***	***-0.03%***	***0.00%***

The call option premium was respectively estimated at $call\ premium = 0.35\%$. The deviation of the numeric parameters from the theoretical values is reasonably small. The deviation is primary due to the fact that in states with low probability, the adjusted probability moves into the negative values zone in which case it is capped by zero value. Capping the adjusted probabilities with zero floor produces an error in mean and standard deviation matching algorithm. In most cases the effect is not material. Overall however the approach showed to be less accurate, less time efficient, and less robust compared to the contraction mapping approach.

G.3 Call option

The section presents standard output of the option valuation tool and discusses how the output should be interpreted.

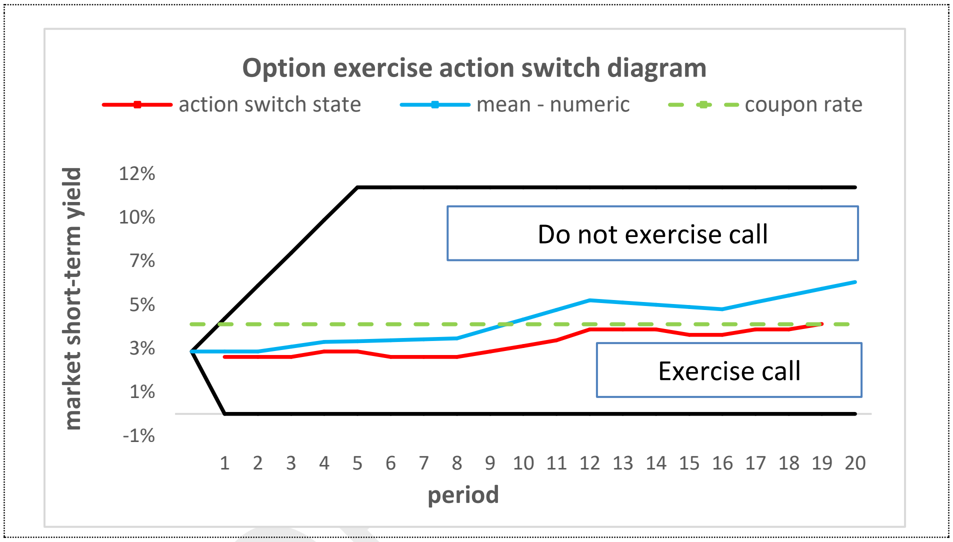
G.3.1 Hull-White (extended Vasicek)

The example discussed in this section was described in Appendix F.3. The example was estimated using the Hull-White (extended Vasicek) model with the following parameters: $\alpha = 0, \sigma = 0.76\%$. The drift parameter was estimated based on the increasing term structure of yield rates with the equivalent constant drift parameter equal to $\vartheta = 0.68\%$. The maturity term was set at 5 years. The credit rating of the underlying bond transaction is Ba3/BB-.

G.3.1.1 Model estimation

The value of the call option premium was estimated at $call\ premium = 0.33\%$. The output of the call option estimation tool is presented in the diagram below.

Exhibit G.3 Output of option estimation tool

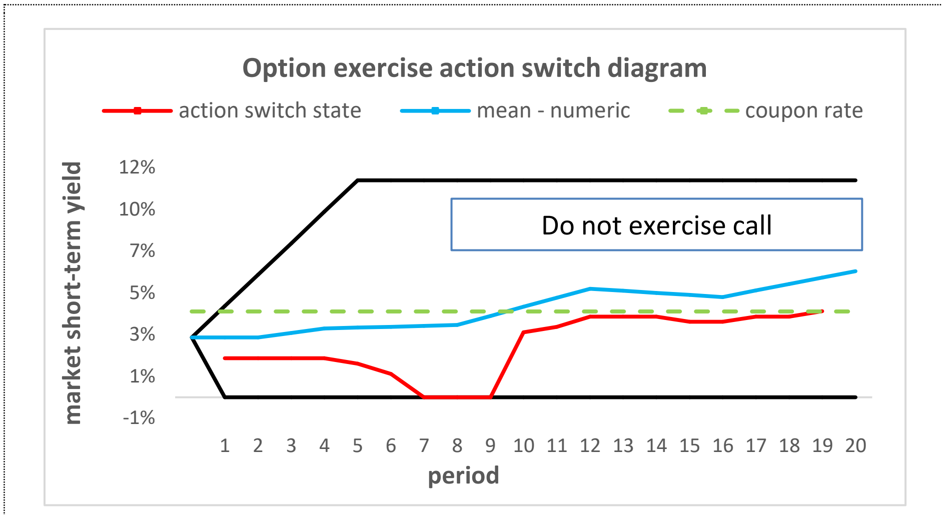


The coupon rate in the model was calibrated at 4.40%, 9 bps lower than the 5-year MYCA rate equal to 4.49%. The coupon rate is shown by the green dashed line in the diagram. The expected short-term rate (shown with the blue line) increases in the model from 3.07% (3-month BB- yield rate reported by Bloomberg) to 6.47%. The increase in the short-term rate is consistent with the increasing term structure of the yield rates. The short-term rate needs to increase above the 5-year BB- yield rate (equal 4.49%) so that the 5-year rate can be replicated as a sequence of increasing short-term rates.

The call option is exercised below the red line. As the bond outstanding term to maturity approaches zero, the bond is exercised when the yield rate is below the bond coupon rate. The black line shows the bounds of the interest rate state set.

Suppose now that the option has a 3% penalty if exercised in the first 2.5 years and zero penalty afterwards. The diagram for the option with the penalty provision is shown below.

Exhibit G.4 Output of option estimation tool



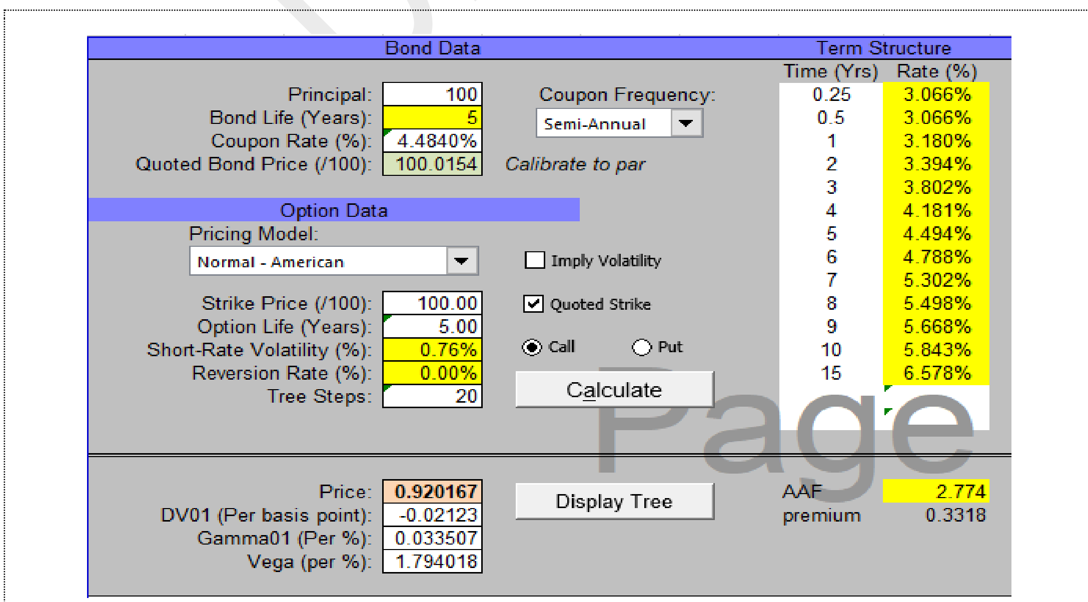
The option premium is reduced to $call\ premium = 0.10\%$. The diagram also shows that the option is never exercised when the penalty is close to expiration. After the penalty is reduced to zero, the diagram becomes the same as the diagram with no penalty.

G.3.1.2 Model validation

Model validation based on the terminal distribution parameters and implied parameters was illustrated for the example in the Appendix F.3. [Zero-coupon price validation] In this section we show how the results are validated directly against the output of the DerivaGem tool.

The output of the DerivaGem tool is presented in the exhibit below. As discussed in Appendix E.1, the coupon rate was calibrated so that the quoted bond price was equal to par.

Exhibit G.5 Validation of option estimation: DerivaGem output

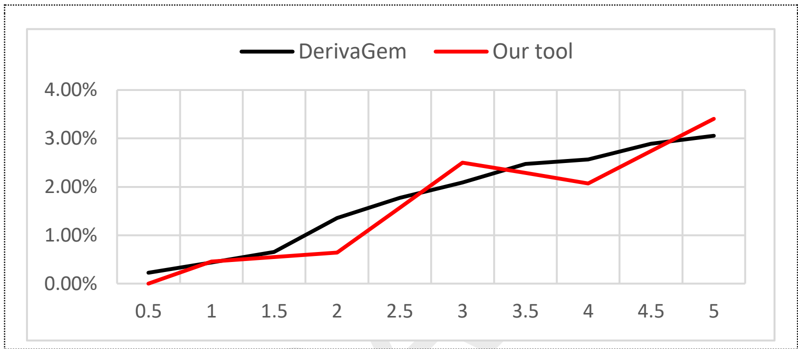


The estimated call option premium $call\ premium = 0.33\%$ matches very closely the premium (0.33%) estimated in our tool and presented in Appendix F.3. We used consistently the DerivaGem tool to validate the results of the option calculation and in each case the difference between the DerivaGem tool and our tool would not exceed 1-2 bps when applied to Vasicek or Hull-White (extended Vasicek) model.

After further review of the DerivaGem output we observed the following facts.

- *Estimated terminal distribution*. The mean and standard deviation of the terminal distribution in DerivaGem and out tool were respectively $\mu = 6.12\%, \sigma = 1.70\%$ and $\mu = 6.47, \sigma = 1.69\%$. We observe certain discrepancy in the estimated mean parameter. The average drift parameter was estimated at $\bar{\vartheta} = 0.61\%$ while in our tool it was estimated at $\bar{\vartheta} = 0.68\%$.
- *Drift parameter*. Estimated drift parameters were different. The diagram that shows two cumulative drift functions estimated in DerivaGem and our tools.

Exhibit G.6 Cumulative drift estimated in DerivaGem and out tool



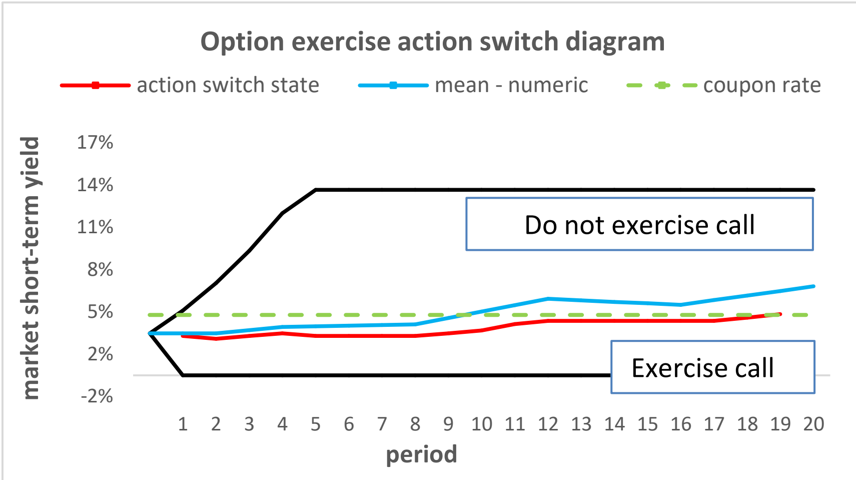
G.3.2 Hull-White (extended CIR)

To compare the result under Hull-White (extended Vasicek) and Hull-White (extended CIR) models, we run the model with the same parameters as described in the previous section but assuming the Hull-White (extended CIR) model specification.

G.3.2.1 Model estimation

Under the Hull-White (extended CIR) model, the estimated volatility parameter is $\sigma = 5\%$. Note that the actual volatility $\sigma_t = \sigma \times \sqrt{r_t}$ ranges between 0.88% and 1.28% along the average path of the interest rates and is higher than the $\sigma = 0.76\%$ estimated in the Hull-White (extended Vasicek) model. The call option premium is also respectively higher and is equal to $call\ premium = 0.26\%$. The call option diagram in the Hull-White (extended Vasicek) model is similar to the diagram in the previous example.

Exhibit G.7 Output of option estimation tool



Due to higher volatility of the interest rates (especially in the area of high interest rates), the bounds of the interest rate states set is wider compared to the state set constructed for the Hull-White (extended Vasicek) model.

To test further comparability of the Hull-White (extended CIR) and Hull-White (extended Vasicek) models, we overrode the historical volatility parameter $\sigma = 5\%$ with a lower value $\sigma = 4\%$ (so that the actual volatility $\sigma_t = \sigma \times \sqrt{r_t}$ ranges between 0.70% and 1.02% along the average path of the interest rates). The call option premium reduced to $call\ premium = 0.19\%$ and was comparable to the call option premium (0.20%) estimated under the Hull-White (extended Vasicek) model.

[Sensitivity to interest rate]

G.3.2.2 Model validation

Note that the DerivaGem tool does not implement the Hull-White (extended CIR) model and therefore cannot be used to validate the estimation output of the Hull-White (extended CIR) model. Therefore, the numerical model can be validated only against the respective theoretical model.

G.4 Put option

The put option estimation is illustrated with the same example of Hull-White (extended Vasicek) model that was described in Appendix F.3.

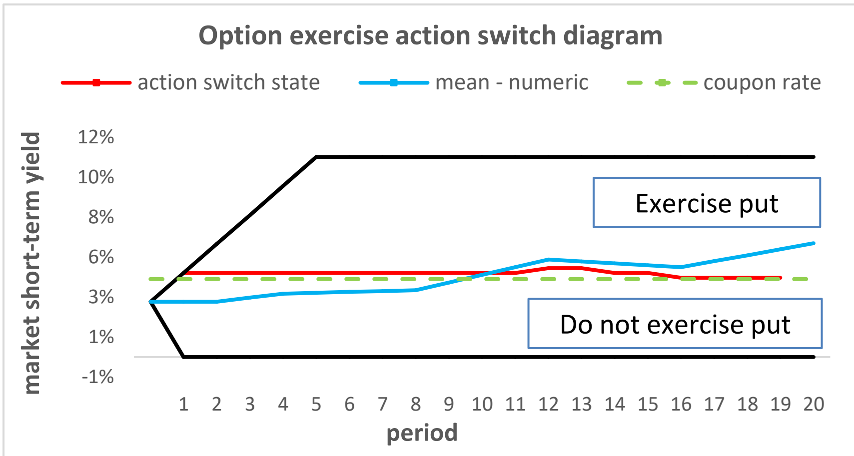
G.4.1 Hull-White (extended Vasicek)

In this section we compare the results of the put option calculation against the results of call option estimation for the Hull-White (extended Vasicek) model.

G.4.1.1 Model estimation

The value of the put option discount was estimated at $put\ discount = 1.10\%$. The output of the call option estimation tool is presented in the diagram below.

Exhibit G.8 Output of option estimation tool

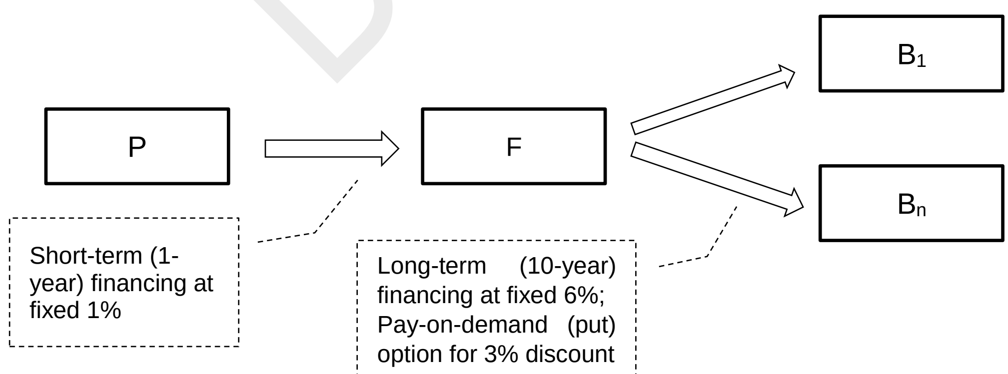


As can be observed from the example, the put option discount is significantly higher than the call option premium. This is typically the case when the term structure of the yield rates is increasing. The term premium between the one-year and five-year rates in the example is 1.22%. Therefore, after adjusting the five-year interest rate for the presence of the put option discount, the term premium component in the five-year rates is reduced from 1.22% to 0.12%. The arbitrage argument, which explains why the put option discount is generally large, is provided in the next section.

G.4.1.2 Put option and term premium arbitrage.

Suppose that F is a financing subsidiary that receives funds from the corporate group parent P and lends them to borrowing subsidiaries Bi. Then the subsidiary F can apply the following lending strategy to generate arbitrage profits (illustrated in the diagram below).

Exhibit G.9 Illustrative example of the term premium arbitrage



To match the funds received from the borrowers Bi and repaid to the parent P, the financing subsidiary F exercises the required number of put options whenever the debt to the parent P is due.

In the example, the financing cost for the subsidiary F is 1% and the interest income is 3% (= 6% - 3%). The profit 2% (= 5% - 3%) is generated due to a risk-free arbitrage produced by the term-premium trading. The example illustrates the following points.

Note! The example has an important implication in transfer pricing analysis. The intercompany loans are often issued as **on-demand** loans without specifying an exact maturity term of the loan. The on-demand term of the loan implies that the repayment of the loan principal amount can be demanded by the lender at any time without any penalties. Because the maturity term of the loan is not specified, it can be treated both as a short-term or a long-term loan. The assumption about the loan maturity will generally have a very material impact on the estimated interest rate (since term premium is one of the key factors that affects the interest rate estimate). However, since on-demand feature on the loan is interpreted as the presence of the put option, the adjustment of the interest rate for the presence of the put option will reduce very materially the term premium component of the interest rate. In practice on-demand loan can be treated as effectively a **short-term loan**.

G.4.1.3 Model validation

The estimated put option discount $put\ discount = 1.09\%$ matches very closely the discount (1.10%) estimated in our tool and presented in Appendix F.3. DerivaGem and our tool produce closely matching put and call option price numbers in the case of Hull-White (extended Vasicek) model.

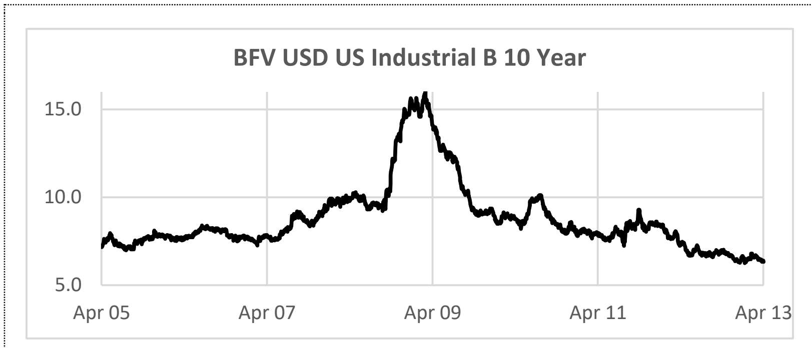
G.5 Debt Refinancing

Example below shows that as the financing cost is decreasing materially for a company, the company has a strong incentive to exercise the call option (even if the prepayment option includes a penalty structure) and to refinance its debt at a lower cost. Therefore, presence of the call option in an intercompany loan agreement can potentially represent a **transfer pricing risk**. Tax authorities may argue that while the interest rate on the loan was at arm's length on the loan issue date, the interest rate may not be at arm's length over the life of the loan if the loan refinancing could result in lower financing costs. Therefore, in the presence of the call option the market interest rates should be monitored on a regular basis to ensure that the borrower does not have an incentive to exercise the prepayment option. Including a penalty structure mitigates partially the loan prepayment transfer pricing risk.

Example describes a debt refinancing history of Compass Minerals International Inc. (**CMP**) based on the respective Bloomberg data. In May 2009 CMP issued a 10-year US$100 million callable senior unsecured note. The call option becomes effective starting in June 2014 (five years after the issue date) with the initial penalty equal to 4% which is reduced then uniformly to zero in 2017 (two years before the note maturity date). The coupon rate on the note is fixed at 8%. The note transaction was rated by Moody's at B+ on the note issue date.

In September 2010 the note was upgraded by Moody's to Ba2 rating. The yield rates on sub-investment debt reached its peak in 2009 and then decreased significantly over time. The history of interest rates (represented by Bloomberg 10-year B rated yield series) is presented in the exhibit below.

Exhibit G.10 History of yield rate movement



As soon as the note becomes callable in June 2014, CMP exercises the call option and repays the note (including the 4% redemption penalty payment). Contemporaneously with debt repayment, CMP issues a new 10-year US$250 million senior unsecured note (with the make-whole provision which termination date is set two months prior to the note maturity date). The coupon rate on the note is fixed at 4.875%. The note transaction was rated by Moody's at Ba2 on the note issue date.

This example illustrates that the following factors affect the prepayment risk.

- ► Significant decrease in the market interest rates;
- ► Improvement in the tested entity creditworthiness;
- ► Decrease in the remaining maturity term of the loan

In the example, the 8% note with five-year remaining effective maturity was refinanced with the 4.875% note with 10-year maturity term.

G.6 Prepayment risk

G.6.1 Prepayment risk adjustment

Prepayment risk may potentially be an important factor, which may affect materially the yield rates on the bonds. An exhibit below illustrates a large variation in the yields on the binds issued by the same entity. The variation is explained by a difference in the bonds coupon rates and respectively large difference in the bonds prepayment risk.

ISIN code	Issuer	Maturity	Next call date	Redemption penalty	Coupon rate	Yield rate[46]
US629377CE03	NRG ENERGY INC	15-Jan-28	15-Jan-23	2.88%	5.750	4.453
US629377CQ33	NRG ENERGY INC	15-Feb-29	15-Feb-24	1.69%	3.375	3.570
US629377CH34	NRG ENERGY INC	15-Jun-29	15-Jun-24	2.63%	5.250	4.235
US629377CR16	NRG ENERGY INC	15-Feb-31	15-Feb-26	1.81%	3.625	3.720
US629377CS98	NRG ENERGY INC	15-Feb-32	15-Feb-27	1.94%	3.875	3.904

In the example, the bonds with higher remaining maturity term have almost 1% lower yield rates. The difference is explained by more than 2% difference in the coupon rates. Prepayment risk pushes the yield

[46] Yield rates were obtained as of 15 October 2021.

rate closer to the coupon rate resulting in the yield discrepancy presented above. In the example, there should be ~1 yield adjustment on the callable yields with high coupon rates.

G.6.2 Exercised callable bonds

G.6.3 Callable bonds with improved credit rating

G.6.4 Callable bonds with decreased remaining maturity term

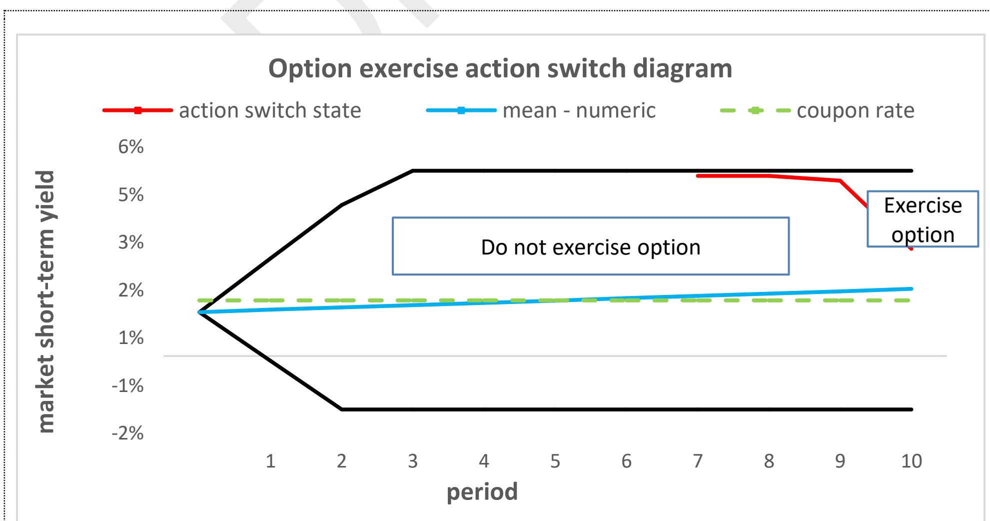
G.7 Short-term loan with an option to lock into a long-term financing

The type of financing may arise, for example, in the case if the borrower needs to secure the funds for short-term purposes but would prefer to have an option to convert it into a long-term financing if necessary. Under the financing type, the objective is to minimize the financing cost by removing term premium from the financing interest rate.

The loan maturity can be set as a long-term maturity with the clause that the interest rate is renegotiated on an annual basis. In the case if the lender and the borrower disagree on the interest rate, the borrower has the option to lock into a long-term financing, The option represents a benefit to the borrower and therefore the short-term interest rate on the loan must be adjusted for the option premium.

The option is modelled in the ac.finance.SRM tool as the loan redemption premium, which is contingent on the current market yield rates. The redemption premium is estimated as the fair market value of the long-term bond. The borrower has incentive to exercise the option whenever the marker rates go up at the 1-year maturity term and the market value of the long-term loan is below the par value.

Exhibit G.11 Option exercise states



(In the example, a short-term loan was modelled using a 1-year fixed 1.7% interest loan (with one year term approximated by 10 discrete periods) and long-term financing was modelled with a 5-year loan with fixed 3.25% interest rate).

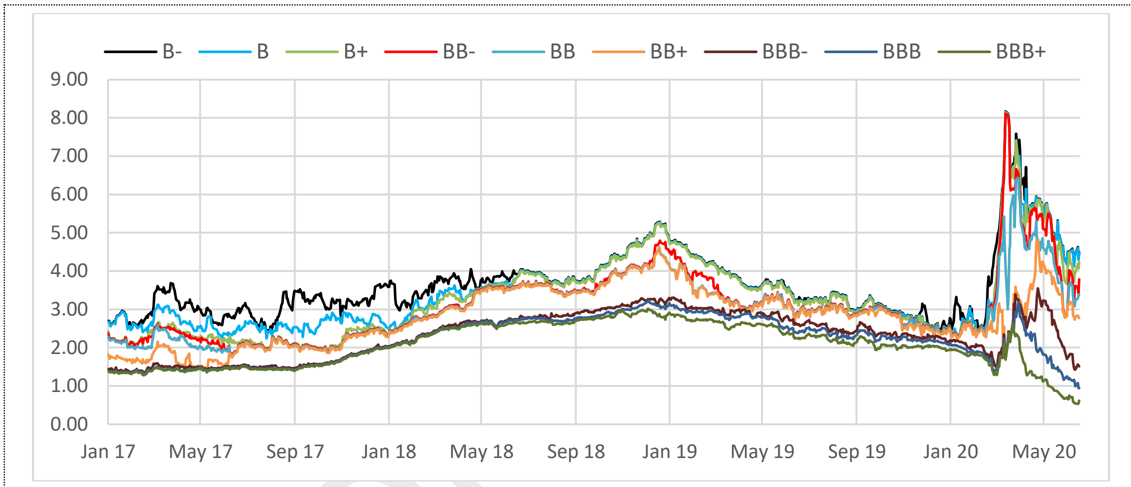
Appendix H Empirical Analysis of Interest Rate Model Parameters

This section reviews the historical behavior of the estimators for the interest rate model parameters. The analysis of historical data was applied in this guide to select the default parameter estimation model (discussed in Section 3).

H.1 Short-term yield rates

The historical behavior of short-term (3-month) Industrial yield rates (as estimated by Bloomberg) is presented in the exhibit below.

Exhibit H.1 Historical behavior of short-term yields

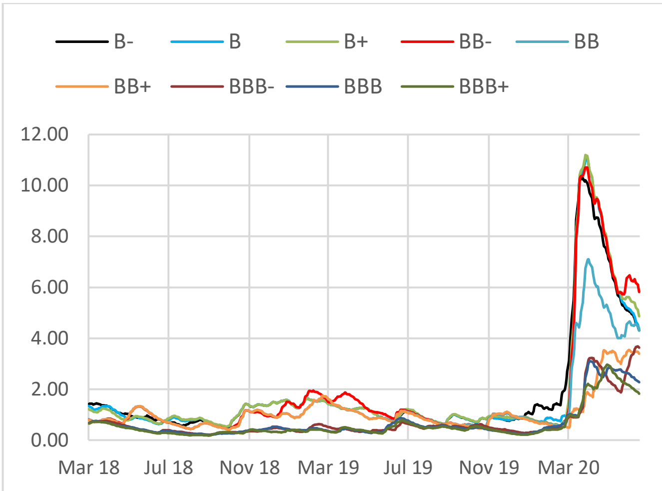


The historical period was selected to include the period of very high market volatility (March-April 2020, the impact of the COVID-9 public health crisis), period of moderately-high in market volatility (January 2019) and periods of relatively low market volatility. The purpose of including periods with different markets behavior is to assess the impact of market conditions on the estimated interest rate model parameters.

H.2 Volatility parameter

The volatility of the market yield rates is presented in the exhibits below. As the first step, we estimate market volatility for the Vasicek model with zero mean reversion parameter. The volatility was estimated based on 1-month changes in the yield rates (normalized to annual volatility) using EWMA model with $\lambda = 0.95$ parameter.

Exhibit H.2 Historical volatility (zero mean-reversion parameter)

<table>
<tr><th>Vasicek model</th><th>CIR model</th></tr>
<tr><td>
</td><td></td></tr>
</table>

The exhibit above shows an extremely high volatility parameter estimate during high market volatility period. This is the result of the random walk model assumption, which produces unrealistic volatility parameter during extreme movements in market interest rates.[47]

The above example illustrates the importance of the mean-reversion assumption in the interest rate modelling.

H.3 Drift parameter

Drift parameter determines the upward slope of the yield rate term structure. Exhibit below shows the estimate of the drift parameter matched to the 10-year slope of the term structure.

Exhibit H.3 Historical drift parameter (zero mean-reversion parameter)

Vasicek model	CIR model

H.4 Mean-reversion parameter

Mean-reversion parameter determines speed of convergence of market interest rates to the long-term equilibrium. In this section, we review the behavior of the mean-reversion parameter under two alternative estimation approaches: (i) mean-reversion parameter calibrated to the long-term equilibrium value; and (ii) mean-reversion parameter estimated using HP filter model.

H.5 Summary

The analysis of historical data was performed to identify the default approach to parameter estimation, which produces reasonable and intuitive results. Specifically,

[47] Random walk model assumes that past extreme behavior in the market rates will also continue in the future.

(i) Drift and volatility parameters change significantly over time and should be estimated as of each specific valuation date to reflect current market conditions. Mean-reversion parameter measures speed of convergence to the long-run steady state and may be viewed as a static parameter which should be updated only periodically (e.g. annually).

(ii) Simple model with zero mean-reversion parameter produces non-usable parameters in the periods of market high volatility. Therefore, mean-reversion and convergence to the steady state must be incorporated into the interest rate model. This is also consistent with the empirical evidence of interest rate historical behavior.

(iii) A standard approach to model a mean-reversion process is to estimate a long-term trend using HP filter and estimate speed of convergence to the long-term trend using simple regression analysis.

(iv) Due to low accuracy of daily data, daily frequency produces non-robust parameter estimates. To improve robustness of the results, the daily data is aggregated into a lower frequency (e.g. monthly) data, which is used in parameter estimation.

Based on the review of the historical data, the following modelling approach and modelling parameters were selected:

1. xx

Appendix I Technical Comments and Modelling Notes

This section presents technical notes which provide a more in-depth insight into the interest rate options modelling.

I.1 Technical Comments

In this section we summarize the technical problems that we encountered while developing and testing the ac.SRM tools.

- **Zero transition probabilities**. This case may be present whenever the model drift parameter is material compared to the model volatility parameter (an example when we observed the problem is $\vartheta = 2\%$ and $\sigma = 0.1\%$). This problem is typical for Hull-White models with variable drift as in certain periods of time the estimated drift parameter may be material. The problem is typically present on the boundary of the set of states. For example, if r_n is the maximum of the discrete set of states and drift parameter is a large positive number compared to the volatility parameter, then the process moves outside the discrete set of states and the probability of each state in the set (estimated based on Normal distribution) is close to a numerically zero value.

  To fix the problem for a state r_t in which the problem was encountered (cumulative probability of child states equals numerically to zero), we assign uniform probabilities to all child nodes with states greater or equal than r_t, if drift is positive and to all child nodes with states less or equal than r_t, if drift is negative.

- **Failure of contraction mapping to converge**. This case may be present in state r_t whenever $r_t + \vartheta_t dt$ is greater (smaller) than the maximum (minimum) state in the discrete set of states. If this problem is encountered, the adjusted transition probabilities assign probability one to the largest (smallest) state in the discrete set of states.

The above points are illustrated by the following example. Suppose that in the example from Appendix G.3, the volatility is overridden by a low volatility value $\sigma = 0.05\%$. Then the distribution over the tree nodes is estimated as follows:

Exhibit I.1 Interest rate tree with low volatility parameter

																			time period		
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2.88%																					
3.07%	100.0%	100.0%	100.0%																		
3.25%				100.0%																	
3.44%					100.0%	100.0%	100.0%	100.0%	100.0%												
3.63%																					
3.82%										23.7%											
4.01%										76.3%											
4.20%											5.6%						0.3%				
4.39%											36.1%					0.3%	4.1%				
4.58%											58.2%	1.3%			0.3%	4.1%	19.6%	0.3%			
4.76%												12.8%		0.3%	4.1%	19.6%	42.1%	4.1%			
4.95%												41.4%	0.3%	4.1%	19.6%	42.1%	33.9%	19.6%	0.3%		
5.14%												44.4%	4.1%	19.6%	42.1%	33.9%	0.0%	42.1%	4.1%		
5.33%													19.6%	42.1%	33.9%	0.0%	0.0%	33.9%	19.6%	0.3%	
5.52%													42.1%	33.9%	0.0%	0.0%	0.0%	0.0%	42.1%	4.1%	
5.71%													33.9%	0.0%	0.0%	0.0%	0.0%	0.0%	33.9%	19.6%	0.3%
5.90%																		0.0%	0.0%	42.1%	4.1%
6.09%																		0.0%	0.0%	33.9%	19.6%
6.27%																			0.0%	0.0%	42.1%
6.46%																			0.0%	0.0%	33.9%

In multiple periods the set of states consist of a single state, which illustrates that at lower volatility values the interest rate process is close to a deterministic process. The theoretical process will generally deviate from the single state and the state probability will be close to zero. The numeric procedure needs to identify those boundary conditions and reassign the probability to one. The transition probability adjustment algorithm based on the contraction mapping also needs to identify that the transition probabilities cannot be readjusted so that to match the theoretical mean value. Therefore, the algorithm does not converge and a unit probability must be assigned to the largest (or lowest) state in the set of discrete states.

I.2 Modelling notes

This section lists assumptions used in the ac.SRM tool implementation.

- **Function inputs for transition probabilities**. Transition probabilities are modelled using two objects: (i) object that models transition probabilities of the one-dimensional states of the process and (ii) object that models the transition probabilities of the process mode. Both evolution of the state and evolution of the mode are described by four numeric values: time t, action (denoted as y for states and a for modes), state or mode in period t (denoted respectively as r_t or m_t) and state or mode in period $t+1$ (denoted respectively as r_{t+1} or m_{t+1}).

  The interest rate process is a special case of a stochastic process with one-dimensional continuous state and a discrete set of modes and mode actions. The process has a blank set of state actions $\{y\}$; two mode actions ($a = 0$ corresponding to not exercising the option and $a = 1$ corresponding to exercising the option), and two modes ($m = 0$ corresponding to the option not being exercised and $m = 1$ corresponding to the option being exercised). The transition probabilities of an interest rate process modes are described by the following mapping (same for each process48):

  $$\begin{cases} (a=0) \Rightarrow \begin{cases} Pr\{(m_t = 0) \Rightarrow (m_{t+1} = 0)\} = 1 \\ Pr\{(m_t = 1) \Rightarrow (m_{t+1} = 1)\} = 1 \end{cases} \\ (a=1) \Rightarrow \begin{cases} Pr\{(m_t = 0) \Rightarrow (m_{t+1} = 1)\} = 1 \\ Pr\{(m_t = 1) \Rightarrow (m_{t+1} = 1)\} = 1 \end{cases} \end{cases}$$

  The transition probabilities of the state are described by default by a function

  $$p(t, y, r_t, r_{t+1}) = \frac{1}{\sqrt{2\pi}\sigma(r_t)} \times e^{-\frac{\left(r_{t+1} - r_t - \mu(t, r_t)\right)^2}{2\sigma^2(r_t)}}$$

- **Transition probability object form**at. Transition probabilities can alternatively be modelled as a composition of function and mapping objects. This input format is applied for example when the transition probabilities are modelled for a trinomial tree, where the up, middle, and down movements of the process are modelled by a respective mapping object.

- **Objective and discount functions**. Objective function in the generic stochastic model is assumed to be a function of the following parameters:

  $$f(t, a, y, m, r)$$

  where t is time, a is mode action, y is state action, m is mode state, and r is action state.49 The function $f(T, a, y, m, r)$ also models the process terminal function. In the case of the interest rate process, the function is estimated as follows:

  - If $m = 1$ then the option was exercised, and the value of the objective function is zero;

48 For a general stochastic process implemented in the tool the transition probabilities of process modes are always assumed to be static (not dependent on time variable).

49 Note that the framework can be applied to model controlled stochastic processes with the state actions (not just mode actions).

► If $m = 0$ and $a = 1$ then the option is exercised, and the value of the function equals the bond redemption value;

► If $m = 0$ and $a = 0$ then the option is not exercised, and the value of the function equals the cash flow paid by the bond in period t.

In the case of the call option, the borrower minimizes the cost of financing and therefore all cash flows described above are assigned a negative value. In the case of the put option, the value of the bond is maximized, and the cash flows are estimated with the positive sign.

The discount function is modelled as a function of the following parameters:

$$D(t, r)$$

Discount depends only on the process state but not the process mode state. Discounts are estimated based on the zero-coupon bond prices described in Sections 2.2 or 2.3

Appendix J References

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[6] www.alexacomputing.com