ESSAYS ON MATCHING:

N-LATERAL MATCHING WITH K DECISIONS

AND

MATCHING WITH COORDINATION FRICTIONS

by

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Abstract

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Ph.D. Thesis (2005) by

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The thesis consists of three essays on matching. In Chapter 2, I introduce an N-lateral K-decision matching model in which a matched group of N (one from each class) must simultaneously choose one of K decisions. For any decision, the surplus generated by a group of matched individuals is linear in the individuals' types. In each of the N classes, I assume that (i) there is a continuum of agents, (ii) the agents are endowed with multi-dimensional types, and (iii) the distribution of the types has a continuous density function. In this case, I show that the equilibrium of the model exists and that it can be constructed by solving an associated convex minimization problem. I provide easily verifiable conditions for the uniqueness of the equilibrium.

In section 3.2 of Chapter 3, I consider a bilateral K-decision matching model in which agents are endowed with one-dimensional types and, in contrast to Chapter 2, the surplus generated in a match may be non-linear. I

provide sufficient conditions under which equilibrium matching is positive assortative. In section 3.4 of Chapter 3, I propose a bilateral stochastic K-decision model in which the utility of each matched individual is a sum of a deterministic component, a match transfer, and an idiosyncratic component. I provide general sufficient conditions for the existence and uniqueness of the equilibrium of the model. The proof of equilibrium existence is constructive.

In chapter 4, I consider a matching model that describes a matching process with coordination frictions in the labor market. In the literature, a standard way to introduce coordination frictions is to describe the market equilibrium as a symmetric equilibrium of a Bayesian game in which workers apply simultaneously to firms and payoffs of the workers at each firm are determined by the Vickrey mechanism. I extend the analysis of the model to the case in which the workers are endowed with multi-dimensional types and there are multiple job positions at each firm. I provide conditions under which the construction of a symmetric equilibrium of the Bayesian game is equivalent to the construction of a solution of a planner's convex constrained optimization problem.

Contents

Chapter 1

Introduction

A matching model is a fundamental analytical tool in the description, design, and empirical evaluation of competitive markets. A matching model is typically applied to the labor, marriage, or housing markets. For example, a marriage market can be modelled as a set of men and women characterized by their types. Men and women are endowed with preferences over the types of the opposite sex and a reserve value of staying unmatched. The following questions are analyzed within the framework of a matching model.

i Does (properly defined) equilibrium of the model exist?

ii What is the structure of the set of equilibria of the matching model?

iii Is there an algorithmic procedure that constructs an equilibrium of the model?

iv What are the incentive properties of the equilibrium of the model?

There are many areas of study in which a matching model is either analyzed directly or as a modelling block of a larger model. As you model different markets you may have to make different assumptions about the matching model that describes the market. For example, you need to choose a matching technology. It is typically one of the following.

1. Random matching. There is no centralized market and people meet randomly in the economy. Each time they meet they have to decide whether to match or not. The model is referred to as a *random matching model*.

2. A single centralized market. In this market you observe and can make an offer to any agent of the opposite type. The model is referred to as a *matching model without coordination frictions*.

3. Several centralized markets. In this case an agent first chooses which market to enter and then he can make an offer to any agent of the opposite type who entered the same market. Matches between agents who entered different markets are not feasible. The model is referred to as a *matching model with coordination frictions*.

In a matching model you also need to assume whether the utility of an agent in a match can be transferred to his partner or not. So a matching market can be modelled as having either

1. *transferrable utility*

2. or *non-transferrable or partially transferrable utility*.

Finally, the matching model can be either

1. *static* (one-period) or

2. *dynamic* (multi-period).

Varying the combinations of these three types of assumptions may vary analytical tools required to analyze the resulting matching model.

My thesis consists of three pieces. In Chapter 2 I introduce an *N-lateral K-decision model* which is a static model with transferrable utility and no coordination frictions. In contrast to the previous literature in which individuals from two classes are matched and matches depend on the types of agents, I consider the model in which (1) individuals from N (where N is an arbitrary finite number) classes are matched in groups (each group contains one agent from each class) and (2) matches depend on both the types of individuals and the decisions made in the matches. In common with the standard matching model, I assume that any match generates some surplus which is allocated among the matched agents. In my model, the surplus generated in a match is linear in the types of the matched agent for any given decision. For this form of the surplus function the transfers in each match are determined only by the decision made in the match. I show that, under certain conditions, the equilibrium exists and is unique, thus extending results from the bilateral matching literature to a multi-lateral model with decisions. I then discuss how, even in the case of bilateral matching models, the model introduced here can address interesting economic questions that could not be addressed using the bilateral models in the previous literature. For example, how a redistribution of income between males and females affects the equilibrium matches and the equilibrium production of public good in the families. In another application I introduce and solve analytically a model in which matches depend on the agents' choices of education level and age to marry. I use the analytical solution to give an example of a comparative statics exercise. I describe how a change in the cost of education affects the matches in the market and the decisions made by the families. This application illustrates also the role of complementarity between male and female choices in the resulting equilibrium matches in the marriage market.

In Chapter 3, I extend the N-lateral K-decision matching model in two directions by relaxing some of the restrictions of the model when $N = 2$. In each extension of the bilateral model, I consider a bilateral matching model with K decisions. In the first extension of the bilateral model, the surplus generated in a match has a more general form than that considered in the first essay. I derive some sufficient conditions and some necessary conditions under which there exists an equilibrium matching which is positive assortative. Specifically, I consider a matching model in which there are two classes of agents, the types of the agents are one-dimensional, and the surplus generated in a match has a general form as a function of the types of the agents and the decision made in the match (in particular, it may be non-separable in the types of the agents). I show that if (1) for any decision the types of the agents are complementary to each other and (2) the type of an agent is complementary to the decisions made in the match, then there exists an equilibrium matching that is positive assortative. In the second extension of the bilateral model, the utility of an agent in a match has a more general form than that considered in the first essay. Specifically, the utility of an agent in a match depends on his type, the type of his partner, decision made in the match, and some random component. I show that, under certain conditions, the equilibrium exists and is unique and I discuss how the equilibrium can be constructed numerically.

In Chapter 4 I introduce a bilateral static matching model with transferrable utility and coordination frictions. The model that I analyze extends previous bilateral matching models with coordination frictions by assuming a more general form of heterogeneity of agents. I consider two solution concepts of the model: (1) an optimal solution of a planner's optimization problem and (2) a symmetric equilibrium of a Bayesian game. I show that the set of symmetric equilibria of the Bayesian game coincides with the set of solutions of the first-order Kuhn-Tucker conditions of the optimization problem. In particular, it follows from the result that the optimal solution of the optimization problem is a symmetric equilibrium of an associated Bayesian game. The Bayesian game interpretation of the model offers two explanations of frictions in the market. The frictions arise in the market because (1) the agents have imperfect information (each agent observes only his own type) and (2) the agents fail to coordinate their choices (agents use symmetric strategies). I show that, under certain conditions, the objective function of the planner's problem is concave so that the symmetric equilibrium of the associated Bayesian game can be constructed numerically using standard convex optimization algorithms.

Chapter 2

N-Lateral K-Decision Matching Model

2.1 Introduction

A one-to-one bilateral matching model describes the way the agents from two classes (for example, buyers and sellers) are matched to each other in such a way that a match consists of one individual from each class (so that for example, one buyer is matched with one seller). Stability has been proposed as the property to be satisfied in any reasonable matching. There always exists a stable matching in a one-to-one bilateral matching model and a stable matching can be constructed by a variety of algorithms proposed in the literature. The one-to-one bilateral matching model and the concept of stability can be naturally extended to a one-to-...-one N-lateral matching model in which there are N classes and a match consists of one individual from each class. In general, however, the set of stable matchings may be empty in the N-lateral matching model. This partially explains why there are few results for these models despite the fact that N-lateral matching models describe many interesting economic situations.

One natural example described by a one-to-...-one N-lateral matching model is the market that consists of three classes of agents: consumers, workers, and firms. The agents are matched in groups of three: one consumer, one worker, one firm. The group makes a decision about which of a finite number of goods to produce.

I propose an N-lateral K-decision model in which groups of N people are matched. When matched, they must choose simultaneously one of a finite number of decisions. For any given decision, the surplus function is linear. The surplus generated in a match is a maximum of such decision dependent surpluses. For convenience, I denote the N-lateral K-decision matching model with this form of the surplus function as $\mathcal{M}_N^{\mathcal{K}}$.

In the case that, in each of the N classes of agents, there is a continuum of agents, each agent is endowed with a multidimensional type, and the distribution of the types of agents has a continuous density function, I show that the model $\mathcal{M}_N^{\mathcal{K}}$ has the following properties.[1]

(i) There always exists an equilibrium of the model. Sufficient easily verifiable conditions can be provided for the uniqueness of the equilibrium.

(ii) The equilibrium can be constructed numerically by solving an associated convex minimization problem.[2]

A variety of matching markets can be naturally modelled by $\mathcal{M}_N^{\mathcal{K}}$. Properties (i) and (ii) allow us to use the model to simulate numerically, and do

[1] I also impose a restriction on the coefficients of the surplus function.

[2] A standard way to construct a stable matching in a bilateral matching problem is to represent it as a linear optimization problem and solve it numerically applying, for example, an auction algorithm (see Roth and Sotomayor, (1989) [30]). In this chapter, I associate with $\mathcal{M}_N^{\mathcal{K}}$ an optimal set partition problem. Each set of the agents' types is partitioned into $K+1$ subsets. One subset of the partition corresponds to the subset of types of unmatched agents and each of the other $k = 1 \dots K$ subsets corresponds to the set of the types of agents who make decision $k = 1 \dots K$ when matched (some of the subsets may be empty). The optimal partition generates naturally the stable matches of groups of agents. The agents from different classes who make the same decision are matched together in groups.

comparative statics of the matching markets.

Even in the special case that $N = 2$, my model offers an alternative way of thinking about how people match and, by extension, an alternative way of defining and constructing the equilibrium in bilateral matching models. This alternative representation of the matching model can provide more insights about how people match.

For example, consider the $N = 2$ lateral, $K = 2$ decision model of the marriage market in which males are matched with females. In the first application, an agent's type is interpreted as income and a decision is a quantity of public good produced within a family[3]. This application (discussed in Section 2.4.1) can address the question of how redistribution of income between males and females affects the quantity of the public good. It has been shown in the literature ([3], [4], [31], [37]) that redistribution of income within a family has no effect on the quantity of the public good produced by the family. However, this literature assumes that redistribution of income does not affect how people match in the marriage market. A simple example, described in this chapter, illustrates that redistribution of income typically affects how people match and, therefore, affects the total quantity of public good produced by families. Since the distribution of income is not neutral in my model, I derive the distributions of income that generate the highest and the lowest total quantity of the public good.

In the second application, an agent's type is interpreted as ability or taste[4]. A decision is interpreted as the education level and age of marriage of each partner. This application illustrates the following points. First, it

[3]More specifically, it can be interpreted, for example, as the number of kids produced within a family.

[4]The difference in the distributions of types between males and females may be either a result of actual physical differences between the two genders or, for example, a result of specific social norms in the society.

provides some intuition on how the existence of complementarity[5] among male and female choices affects the matches and which decisions are made in equilibrium. Second, some interesting comparative static exercises can be done using the model that may explain how a change in the cost of education for females affects the education and age of each partner in a match.

2.2 Literature

In this section, I discuss some known results of the bilateral matching literature.

In a standard bilateral matching model with transferrable utility and a finite number of agents, individuals from two different classes, say males and females, are matched in a one-to-one fashion. Let's index males with $i \in \mathcal{I}$ and females with $j \in \mathcal{J}$. If a male i and a female j are matched, they generate some surplus u^{ij}. Matching is defined as (i) a one-to-one matching function $m : \mathcal{I} \to \mathcal{J} \cup \emptyset$ that describes the way the agents match ($m(i) = \emptyset$ is interpreted as male i stays unmatched), and (ii) transfer functions, p_1^i and p_2^j, that correspond to each agent to a transfer that the agent obtains ($p_1^i + p_2^{m(i)} = u^{i,m(i)}$). Equilibrium of the model is defined as a matching function $m(\cdot)$ and transfer functions p_1^i and p_2^j such that some stability conditions hold. (Matching is stable if (i) the sum of transfers of any two matched agents equals the surplus generated in the match and (ii) the sum of transfers of any two arbitrary agents is greater or equal to the surplus generated in the match.) Equilibrium always exists in a bilateral matching model (formal definitions and results can be found, for example, in Roth, Sotomayor, (1989) [30]).

[5]Complementarity among choices is a restriction on the surplus function and is defined formally in Section 2.4.2

If there is a continuum of individuals, then they can not be indexed as in the previous model. Instead, it is assumed that each male and female is endowed with a type denoted by $x_1 \in X_1$ and $x_2 \in X_2$ and the mass of males and females of a given type is described by distributions μ_1 and μ_2 correspondingly. The surplus generated by a matched pair of a male of type x_1 and a female of type x_2 is a function of the types of the individuals denoted by $u(x_1, x_2)$. Matching is defined as (i) a matching function, $m : X_1 \to X_2 \cup \emptyset$ that describes the way the agents match ($m(x_1) = \emptyset$ is interpreted as a male of type x_1 stays unmatched) and satisfies some mass balance condition[6] (which is a continuum analogue of the one-to-one mapping condition in the finite case), and (ii) transfer functions, $p_1(x_1)$ and $p_2(x_2)$, that corresponds to each type a transfer that an agent of a given type obtains ($p_1(x_1) + p_2(m(x_1)) = u(x_1, m(x_1))$).

If the types x_1 and x_2 are one-dimensional ($x_n \in \mathbb{R}$), then agents can be ordered with respect to their types. Becker, (1973) [2] shows that under certain conditions the equilibrium matching is positive assortative [7].

When the types of the agents are multi-dimensional, a closed-form solution to the matching problem typically does not exist [8]. In this case, the equilibrium of the model can be constructed numerically, for example, by an auction algorithm[9]. The algorithm is a procedure, which is very similar

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[6]Formally, the mass balance condition is described as follows. For any set of female types E, the set of male types that are matched with some type in E has the same measure as E.

[7]Matching is positive assortative if the matching function from the set of male types into the set of female types is non-decreasing.

[8]An exception to this is a linear quadratic matching model in which there is a continuum of agents endowed with multi-dimensional types, the surplus function is quadratic $u(x_1, x_2) = x_1^T U x_2$, and the distribution of types in each class is a multi-dimensional normal distribution. The surplus maximizing matching can be shown to be linear, $x_2 = m_0 + M x_1$, and explicit expressions for m_0 and M can be derived.

[9]For the description of auction algorithms see Roth, Stomayor (1989) [30] or Bertsekas (1989) [6]. The algorithm is applicable only if the number of agents' types is finite.

to an ascending bid auction. For some special forms of the surplus function the equilibrium matching can be described by a system of partial differential equations (see, for example, [11]).

In general, the equilibrium matching function $m(\cdot)$ may not exist. To obtain existence of the equilibrium matching a more general definition of matching is used in the literature. In the definition, matching is described by (i) a measure μ defined on the direct product of the sets of agents' types ($\mu(A \times B)$ denotes the mass of matches of a male and a female in which the male's type belongs to A and the female's type belongs to B) and (ii) transfers of the matched individuals p_1x_1 and p_2x_2. Gretsky, Ostroy, Zame, (1999) [14] show that equilibrium exists and provide sufficient conditions for the uniqueness of the equilibrium of the model (the equilibrium is generically unique). These conditions, however, are not easily verifiable in practice.

In the cited literature, agents do not make any decisions and the surplus function depends only on the types of the agents. Cole, Malaith, and Postlewaite ([8]) analyze a bilateral matching model in which agents from each side of the market are endowed with one-dimensional types and choose from among a continuum of one-dimensional investment decisions before they match. The investment decisions made by one side of the market are complementary to those made by the other side of the market. When matched, each pair of agents bargain over the division of the surplus generated. The assumptions guarantee assortative matching and allow the authors to focus on conditions that prevent the hold-up problem. The restrictive assumptions that the authors make narrows the number of applications of their model. In particular, the model can not be used in the applications and can not address comparative static questions that I discuss in this chapter. By contrast, I analyze an N-lateral, K-decision matching model in which agents from each side of the market are endowed with multi-dimensional types and choose from among a finite number of multi-dimensional investment decisions. The decision is made at the same time as agents match. Alternatively, as in the hedonic pricing literature (see, for example, [28]), I may assume that agents make decisions before they match and that agents have rational expectations about the transfers they obtain as a function of their decision.

2.3 Matching Model

In this section, I formally describe the matching model $\mathcal{M}_N^{\mathcal{K}}$ and define the equilibrium of the model.

2.3.1 Model Set-Up

There are N classes of agents and there is a continuum of agents in each class. Each agent is endowed with a multi-dimensional type. The type x_n of an agent from a class n is an element of the set $X_n = \mathbb{R}^{\eta_n}$. Let μ_n denote the distribution of types of the agents in class n. The distributions μ_n are assumed to have a *continuous density function*[10], denoted by $f_n(x_n)$.

If a group of N agents, one from each class, match together they generate a surplus

$$u\left(x_1, \ldots x_N\right) = \max_{k \in \mathcal{K}} \left(\sum_{n=1}^{N} a_n^k x_n + a_0^k \right) \tag{2.3.1}$$

where x_n are the types of the agents in the group, $\mathcal{K} = \{0, 1, \ldots, K\}$, $a_n^k \in \mathbf{R}^{\eta_n}$ are some vectors of coefficients, a_0^k are some scalars, and $a_n^k x_n = \sum_{j=1}^{\eta_n} a_n^{k,j} x_n^j$. The set $\mathcal{K}$ is interpreted as the set of decisions so that if a

[10]Though it may be true that a continuous density is not required to prove the existence of an equilibrium when there is a continuum of agents, equilibrium may not exist when continuity fails and there is a finite number of agents.

matched group of N people chooses a decision k from the set $\mathcal{K}$ they generate the linear surplus

$$u^k(x_1, \dots x_N) = \sum_{n=1}^{N} a_n^k x_n + a_0^k \tag{2.3.2}$$

When a group of agents match they choose a decision k that generates the highest surplus over all decision dependent surpluses u^k. Decision $k = 0$ is interpreted as a decision to stay unmatched. In this case the agent of type x_n from class n in the matched group obtains his reserve value

$$r_n(x_n) = a_n^0 x_n + r_n^0 \tag{2.3.3}$$

where r_n^0 are some constants ($\sum_n r_n^0 = a_0^0$). I show in section 2.3.2 that if the functions $u^k(x_1, \dots, x_N)$ are linear then in equilibrium each individual cares only about the decision made by the matched group of agents but not about the types of the other agents in the group [11].

The way the agents are matched is described by a vector of functions $m = (m_2(\cdot), \dots, m_N(\cdot))$, where $m_n : X_1 \to X_n \cup \emptyset$ (I interpret $m_n(x_1) = \emptyset$ as an agent of type x_1 stays unmatched). The vector of functions $m = (m_2(\cdot), \dots, m_N(\cdot))$ satisfies the following two conditions.

1. If $m_n(x_1) = \emptyset$ for some n then

$$m_n(x_1) = \emptyset \tag{2.3.4}$$

for any n

2. For each n

$$\mu_n(E) = \mu_1(m_n^{-1}(E)) \tag{2.3.5}$$

for any $E \in X_n$ measurable with respect to μ_n.

[11] This is true in a more general case when the functions $u^k(x_1, \dots, x_N)$ are separable in agents types.

I note that each group of matched agents contains exactly one agent from each class. The first condition guarantees that if $m_n(x_1) = \emptyset$ for any n, then type x_1 agents are unmatched; otherwise, type x_1 agents are matched with agents of types $(m_2(x_1), \ldots, m_N(x_1))$. The second condition is a *mass balance* condition[12].

2.3.2 Definition of Equilibrium

If a group of agents with types $(x_1, \ldots, x_N)$ is matched together and chooses decision k, then the surplus they generate, described in equation 2.3.2, is allocated among the agents in such a way that an agent of type x_n obtains a portion of surplus, denoted by $p_n^k(x_n)$. I assume that utility equals the portion received. I restrict[13] to allocations for which the portion of the surplus received by agents of type x_n is of the form

$$p_n^k(x_n) = a_n^k x_n + p_{n,0}^k \tag{2.3.6}$$

where $p_0 = \left\{p_{n,0}^k : n = 1, \ldots, N,\ k = 0, \ldots, K\right\}$ is an array of unknown transfer parameters and $\sum_n p_{n,0}^k = a_0^k$. If $k = 0$ then I assume that

$$p_{n,0}^0 = r_n^0 \tag{2.3.7}$$

That is, the transfer of an unmatched agent equals to the constant parameter in his reserve value function. With this form of surplus allocation, an increase in an agent's type generates an increase in the portion of the surplus received by the agent that is equal to the agent's marginal contribution to the surplus

[12]The conditions 2.3.4 and 2.3.5 do not imply that matching functions $m_2(\cdot), \ldots, m_N(\cdot)$ are one-to-one in a measure theoretic way. That is, there may exist types $x_1 \in X_1$ and $x_1' \in X_1$ such that $x_n = m_n(x_1) = m_n(x_1')$ for some n. This is possible because there is a continuum of agents of each type. Therefore, it is feasible that some agents of type x_n are matched with agents of type x_1 and some with agents of type x_1'.

[13]As I show later in this section there is no loss of generality in this restriction. Any stable matching in this model corresponds to some equilibrium defined below.

generated in a match. The transfers $p^k_{n,0}$ determine only how the constant term a^k_0 is divided in a match if decision k is made.

Let $p_{n,0} = \left\{p^k_{n,0} : k = 1 \dots K\right\}$ be the vector of transfers. The *upper demand set* $\overline{D}^k_n(p_{n,0}) \in X_n$ for decision k in set X_n is defined as

$$\overline{D}^k_n(p_{n,0}) = \left\{x_n \in X_n : p^k_n(x_n) \geq p^l_n(x_n) \quad \text{for any} \quad l = 0 \dots K\right\} \tag{2.3.8}$$

The *lower demand set* $\underline{D}^k_n(p_{n,0}) \in X_n$ for decision k in set X_n is defined as

$$\underline{D}^k_n(p_{n,0}) = \left\{x_n \in X_n : p^k_n(x_n) > p^l_n(x_n) \quad \text{for any} \quad l = 0 \dots K\right\} \tag{2.3.9}$$

and *a demand set* $D^k_n(p_{n,0}) \in X_n$ for decision k in set X_n is any set such that

$$\underline{D}^k_n(p_{n,0}) \subseteq D^k_n(p_{n,0}) \subseteq \overline{D}^k_n(p_{n,0}) \tag{2.3.10}$$

The set $D^0_n(p_{n,0})$ is the subset of types, such that an agent from class n of type $x_n \in D^0_n$ stays unmatched and obtains his reserve value $r_n(x_n)$.

The demand sets $\overline{D}^k_n(p_{n,0})$ and $\underline{D}^k_n(p_{n,0})$ that correspond to a decision k is a set of vectors that are solutions to a linear system of inequalities. Such sets are called *convex polyhedrons*. In particular, the sets are convex and connected.

Now I define the equilibrium of the model[14].

Definition 2.3.1 *A matrix of transfers $p^k_{n,0}$ and corresponding demand sets $D^k_n(p_{n,0})$, defined by equations 2.3.8-2.3.10, describe an equilibrium of $\mathcal{M}^{\mathcal{K}}_N$ if:*

1. *For each n, the collection of the sets $\{D^k_n\}_{k=0,\dots,K}$ generates a partition of the set X_n. That is,*

$$X_n = \sqcup^K_{k=0} D^k_n \qquad \textit{and} \qquad \mu_n(D^k_n \cap D^{\widetilde{k}}_n) = 0 \tag{2.3.11}$$

   for any $k \neq \widetilde{k}$.

[14] An alternative equivalent definition is given in section 2.6.5.

2. *For each $k \neq 0$, the measure of the agents that prefer decision k is the same in each set X_n. That is, for each $k = 1, \ldots, K$*

$$\mu_n \left(D_n^k\right) = \mu_{\tilde{n}} \left(D_{\tilde{n}}^k\right) \tag{2.3.12}$$

for any $n, \tilde{n}$.

3. *The sum of transfers to the agents in a matched group equals the constant term in the surplus generated by the group. For each $k = 1, \ldots, K$*

$$\sum_n p_{n,0}^k = a_0^k \tag{2.3.13}$$

Condition 2.3.12 is a *mass balance* condition. It ensures that an equal mass of agents from each class prefer any given decision $k = 1, .., K$. Condition 2.3.13 is a *surplus balance* condition. It ensures that the sum of the surplus portions allocated to each agent in a match equals the surplus generated in the match.

An equilibrium generates matched groups of N individuals, one from each class. All agents in a match choose a common decision k. Each match consists of one agent from each class chosen arbitrary from a set of agents who choose a common decision k.

In the next proposition I show that equilibrium defined in 2.3.1 generates a stable matching. But, first, I give the definition of a stable matching.

Definition 2.3.2 *A matching $m = (m_2(x_1), \ldots, m_N(x_1))$ is stable if there exist transfers $p_n(x_n)$, $n = 1, \ldots, N$ such that for any vector of types $(x_1, \ldots, x_N)$*

$$p_1(x_1) + p_2(x_2) + \ldots + p_N(x_N) \geq u(x_1, \ldots, x_N) \tag{2.3.14}$$

and for any matched group $(x_1, m_2(x_1), \ldots m_N(x_1))$

$$p_1(x_1) + p_2(m_2(x_1)) + \ldots + p_N(m_N(x_1)) = u(x_1, \ldots, x_N) \tag{2.3.15}$$

The following statement shows the relationship between stable matching and equilibrium of the model

Proposition 2.3.1

(i) Equilibrium matching and transfers generate a stable matching[15].

(ii) Any stable matching is generated by some equilibrium matching and transfers.

A corollary of the proposition is that there does not exist a stable matching in which the transfer functions are not linear. Therefore, constraining the transfers to the functional form 2.3.6 does not reduce the set of stable matchings of the model.

2.3.3 Optimization Problem

In a standard model of supply and demand, equilibrium can be constructed as a solution of an associated planner's optimization problem. Let's consider, for example, a market for a homogeneous good in which inverse demand and supply functions are $p_d(Q)$ and $p_s(Q)$, where Q is the quantity of the good. The planner chooses Q so as to maximize the social surplus function which is defined as $W(Q) = \int_0^Q \left[p_d(\widetilde{Q}) - p_s(\widetilde{Q}\right] d\widetilde{Q}$. The first-order condition to the planner's optimization problem is $p_d(\widehat{Q}) = p_s(\widehat{Q})$. The surplus maximizing values of $\widehat{Q}$ and $p_d(\widehat{Q})$ are interpreted as the equilibrium quantity and price in the model. The condition $p_d(\widehat{Q}) = p_s(\widehat{Q})$ is interpreted as demand equals supply at the equilibrium price. In this chapter, I also construct an equilibrium of the model by finding a solution of an associated planner's optimization problem. There is no analogue, however, between $\mathcal{M}_N^{\mathcal{K}}$ and the simple model

[15]Formally, this means that the set of matches and the transfers to the agents in the equilibrium are the same as the set of matches and the transfers in the stable matching.

of a commodity market described above. In particular, I can not associate directly the equilibrium of the model with the first-order conditions of the planner's problem. Instead, I construct demand sets and transfers by solving a planner's problem that involves demand sets, transfers, and mass balance condition. I begin by introducing the optimization problem in which the planner finds matching functions $m_2(\cdot), \ldots, m_N(\cdot)$ that maximize aggregate surplus. I then introduce the planner's problem in which the planner finds the optimal partition of the sets of types among all partitions that satisfy mass balance condition. The dual to this problem is the one in which the planner finds the transfers that give rise to the demand sets which satisfy mass balance condition. Finally, I describe in Theorem 2.3.1 the relationship between the planner's surplus maximization problem, the dual problem, and the equilibrium of the model.

The planner's natural surplus maximization problem is described as follows. Let, for any given matching functions $(m_2(\cdot), \quad m_N(\cdot))$, the aggregate surplus be defined as the sum of the reserve values of the unmatched agents and the sum of surpluses of the matched groups of agents. The planner chooses the matching functions that maximize the aggregate surplus.

$$\begin{cases} \max_{m_2(x_1),\ldots,m_N(x_1)} \int_{X_1 \setminus \Pi_1^0} u\left(x_1, m_2(x_1), \ \ldots, m_N(x_1)\right) f_1(x_1) dx_1 \\ \qquad\qquad\qquad\qquad + \ \sum_n \int_{\Pi_n^0} r_n(x_n) f_n(x_n) dx_n \\ \\ \text{s.t.:} \quad \text{for each } n \quad \mu_n(E) = \mu_1(m_n^{-1}(E)) \\ \qquad\qquad\qquad\qquad \text{for any } E \in X_n \quad \text{measurable with respect to } \mu_n \end{cases} \tag{2.3.16}$$

where, for each n, the set Π_n^0 is the subset of the types of unmatched agents in class n and the constraint in the planner's problem is the stability condition 2.3.5 that any matching function must satisfy. Figure 2.3.1 illustrates a matching between the types of males and females.

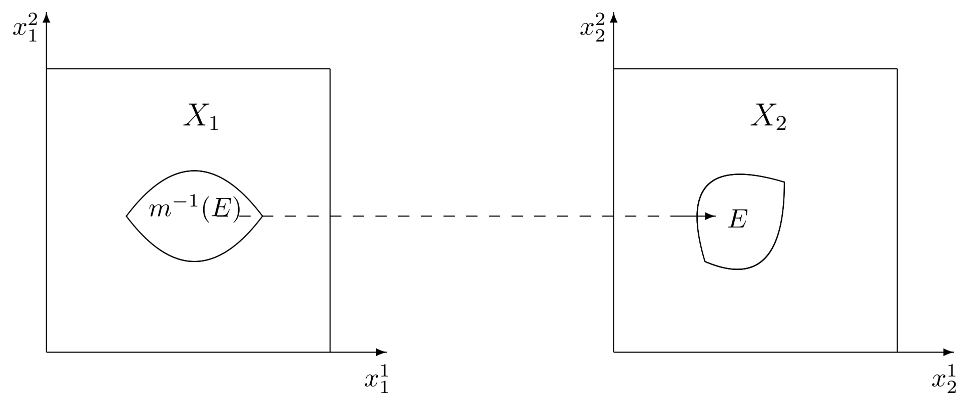


Figure 2.3.1 There are two classes of agents who are endowed with two-dimensional types. A sample distribution shows the distribution of the types in each class. Function $x_2 = m_2(x_1)$ describes the matching between the types of the agents. The matching function $m(x_1)$ satisfies mass balance condition 2.3.5 so that for an arbitrary set $E \in X_2$ the number of the agents' types that belong to the set E equals to the number of the agents' types in set X_1 that belong to the set $m^{-1}(E)$ of types in $\in X_1$ that is mapped into E by $m(x_1)$.

Now I show that the surplus maximization problem can be equivalently represented as a problem in which the planner chooses an optimal partition of the sets of types among all the partitions that satisfy mass balance condition. Matching functions $m(\cdot) = (m_2(\cdot), \ldots, m_N(\cdot))$ generate a natural partition of each set X_n into $K+1$ sets where each type in the set makes a given decision. The partition of a set X_n is described as follows. The set Π_n^0 denotes those types of agents x_n for whom $x_n \neq m_n(x_1)$ for any x_1. That is, the set Π_n^0 consists of types of agents who remain unmatched. The set Π_n^k consists of those types of agents $x_n = m_n(x_1)$ for whom decision $k = 1, ..., K$ generates the largest surplus conditional on the matching $m(x_1)$. That is,

$$a_1^k m_n^{-1}(x_n) + \sum_{i=2}^N a_i^k m_i(m_n^{-1}(x_n)) + a_0^k \geq$$
$$a_1^l m_n^{-1}(x_n) + \sum_{i=2}^N a_i^l m_i(m_n^{-1}(x_n)) + a_0^l$$

for any $l = 0, \dots, K$. Note that the fact that the surplus function $u^k(\cdot)$ is separable in types implies that if two matchings generate a given partition of the sets of types $\Pi_n^k, k = 0, ...K$ into decision sets, then the two matchings generate the same total surplus.

Let $\mathcal{P}_n$ denote partition of set X_n into subsets Π_n^k, $k = 0, \dots, K$ so that $X_n = \left(\sqcup_{k=1}^K \Pi_n^k\right) \sqcup \Pi_n^0$. Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_N)$ denote the vector of partitions. I consider the following total surplus function that maps the set of all partitions into the real line.

$$V(\mathcal{P}) = \sum_{k=1,\dots,K} \left[\sum_{n=1\dots N} \int_{x_n \in \Pi_n^k} a_n^k x_n f_n(x_n) dx_n + a_0^k \mu_1(\Pi_1^k) \right] + \sum_n \int_{x_n \in \Pi_n^0} r_n(x_n) f_n(x_n) dx_n \quad (2.3.17)$$

The optimization problem can now be reformulated as follows. Find the vector of partitions $\mathcal{P}$ that maximizes the following function

$$\begin{cases} \max_{\mathcal{P}} V(\mathcal{P}) \\ \text{s.t. for each } k \quad \mu_1^k(\Pi_1^k) = \mu_n^k(\Pi_n^k) \text{ for any } n \end{cases} \quad (2.3.18)$$

In the optimization problem Π_n^k, $k = 1, \dots, K$ is a set of agents in X_n who make decision k, when matched, and Π_n^0 is a set of unmatched agents in the set X_n.

It is not clear how to solve for the optimal partitions of the sets[16]. Instead

[16]It is not clear, for example, how to differentiate function $V(\mathcal{P})$ with respect to $\mathcal{P}$.

of solving optimization problem 2.3.18 directly I formulate the dual problem. Let

$$W_n\left(p_{n,0}\right) = \sum_{k=1}^{K} \int_{D_n^k(p_{n,0})} \left[a_n^k x_n + p_{n,0}^k\right] f_n(x_n) dx_n + \int_{D_n^0(p_{n,0})} \left[a_n^0 x_n + p_{n,0}^0\right] f_n(x_n) dx_n \quad (2.3.19)$$

where for each n, (1) the sets D_n^k satisfy condition 2.3.10 for any $k = 0, \ldots, K$ and (2) the collection of sets $\{D_n^k\}_{k=0,\ldots,K}$ generates a partition of the set X_n. That is, $X_n = \sqcup_{k=0}^{K} D_n^k$ and $\mu_n(D_n^k \cap D_n^{\widetilde{k}}) = 0$ for any $k \neq \widetilde{k}$. Let also

$$W(p_0) = \sum_n W_n(p_{n,0}) \quad (2.3.20)$$

where $p_0 = (p_{n,0}),\ n = 1 \ldots N$. The dual optimization problem is described as follows. Find an array of values p_0 that is a solution to the following minimization problem.

$$\begin{cases} \min_{p_0} W(p_0) \\ \text{s.t. } \sum_n p_{n,0}^k = a_0^k \quad \text{for any } k = 1, \ldots, K \\ p_{n,0}^0 = r_n^0 \end{cases} \quad (2.3.21)$$

A solution p_0 to the problem 2.3.21 generates demand sets D_n^k, defined in 2.3.10, that partition the sets of types into the subsets of agents who choose different decisions. In the following lemma I provide conditions under which the equilibrium of the model exists and the demand sets that correspond to the optimal array of transfers $p_{n,0}^k$ generate the optimal partition of the sets X_n.

Theorem 2.3.1 *The following statements hold.*

(i) *For any array of transfers* p_0 *that satisfies the constraints of the problem 2.3.21 and for any partition vector* $\mathcal{P}$ *that satisfies the constraints of the problem 2.3.18 the value of the dual objective function is greater or equal to the value of the objective function of the optimal partition problem, that is* $W(p_0) \geq V(\mathcal{P})$.

(ii) *Suppose that Assumption 2.3.1 holds and in each class there is a continuum of agents endowed with multi-dimensional types and the distribution of the types has a continuous density function. Then the optimal array of transfers* $\widehat{p}_0$ *generates the partition* $\mathcal{P}(\widehat{p}_0)$ *of the sets* X_n *into demand sets* D_n^k, *defined as in 2.3.10, such that* $W(\widehat{p}_0) = V(\mathcal{P}(\widehat{p}_0))$. *Partition* $\mathcal{P}(\widehat{p}_0)$ *is a solution to the problem 2.3.18. The transfers* $\widehat{p}_0$ *and the corresponding demand sets* D_n^k *describe the equilibrium of the model.*

(iii) *If* $W(\widehat{p}_0) > V(\widehat{\mathcal{P}})$, *where* $\widehat{p}_0$ *is a solution of the dual problem 2.3.21 and* $\widehat{\mathcal{P}}$ *is a solution of the primary problem 2.3.18 then the equilibrium of the model does not exist.*

Proof is in the Appendix. To summarize, the problem is modified so that instead of looking for the optimal matching functions I look for the array of transfers that generates an optimal partition of the sets X_n. The optimal matching is constructed from the optimal partition as follows. If an agent belongs to a set D_n^0 then he stays unmatched. For each decision $k = 1, \ldots, K$ the agents that belong to the sets D_n^k, $n = 1 \ldots N$, are matched to each other in an arbitrary way. This can be illustrated by the following picture.

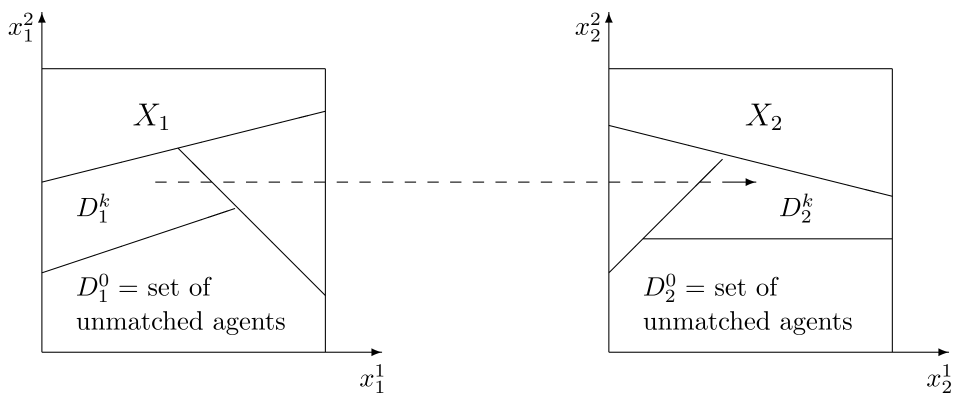


Figure 2.3.2 There are two classes of agents who endowed with 2-dimensional type. An array of optimal transfers p_0 generates a partition of the set X_n of types of the agents into the sets D_n^k such that $\mu_1^k(D_1^k) = \mu_2^k(D_2^k)$. The optimal matching maps the agents' types from a set D_1^k into the set D_2^k for each k in an arbitrary way.

2.3.4 Properties of the Optimization Problem and Existence of the Equilibrium

In this section, I prove that the equilibrium of the model exists and show how the equilibrium can be constructed numerically. First, I impose some restrictions on the coefficients a_n^k under which the objective function in optimization problem 2.3.21 is differentiable. Then I show in Proposition 2.3.2 that existence of equilibrium follows directly from the differentiability of the objective function. In Theorem 2.3.2 I generalize Proposition 2.3.2 and show that the equilibrium of the model exists for an arbitrary array of coefficients a_n^k. I derive then the second derivative of the objective function of the optimization problem 2.3.21. I use the second derivative to show that the objective function is convex[17] and, therefore, the problem 2.3.21 can be solved by standard numerical methods.

[17]It can be shown that the objective function is convex under very general conditions. In particular, convexity of the function can be proved in the cases when the function is not differentiable and even in the cases when the equilibrium of the model does not exist.

To derive the first and the second derivatives of the objective function in the optimization problem 2.3.21 I impose the following restriction on the coefficients a_n^k.

Assumption 2.3.1 *For each n,*

$$a_n^k \neq a_n^l \tag{2.3.22}$$

for any $k \neq l$[18].

Note that if Assumption 2.3.1 holds, then the set of types of agents who are indifferent among several decisions has zero measure. Therefore, $\mu_n(\overline{D}_n^k \cap \underline{D}_n^k) = 0$ and demand set $D_n^k(p_{n,0})$ can be chosen, for example, as the upper demand set[19].

To show that the equilibrium of the model exists, I find the first derivative of $W_n(p_{n,0})$, defined in 2.3.19, with respect to $p_{n,0}^k$. Note, first, that variable $p_{n,0}^k$ is present both in the integrands and the sets $D_n^l(p_{n,0})$, $l = 0, \ldots, K$ over which the integrals is taken. First, I look at how each set $D_n^l(p_{n,0})$ is changing as $p_{n,0}^k$ increases to $p_{n,0}^k + \epsilon$, $\epsilon > 0$. The change in the sets of integration is illustrated in Figure 2.3.3.

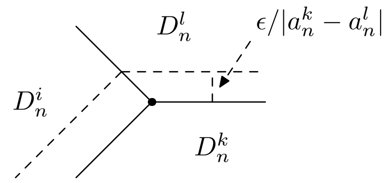


Figure 2.3.3 A change in demand set D_n^k and D_n^l, $l \neq k$, as the transfer $p_{n,0}^k$ of decision k in the set X_n increases by $\epsilon > 0$. Each boundary of the set D_n^k with D_n^l shifts away from D_n^k in a parallel fashion.

[18]The assumption is not so restrictive since the set of parameters that satisfies this assumptions is dense in the original set of parameters $\mathbf{R}^{\eta_1} \times \ldots \times \mathbf{R}^{\eta_N}$.

[19]Assumption 2.3.1 also guarantees that demand sets change continuously with transfers.

Let Γ_n^{kl} denote the boundary between the sets D_n^k and D_n^l :

$$\Gamma_n^{kl} = \Big\{ x_n \in X_n : p_n^k(x_n) = p_n^l(x_n), \\ p_n^k(x_n) \geq p_n^i(x_n) \text{ for any } i = 0, \ldots, K \Big\} \quad (2.3.23)$$

Let $\int_{\Gamma_n^{kl}} h(x_n) f_n(x_n)$ denote the surface integral of a function $h(x_n)$ over the boundary Γ_n^{kl} in R^{M_n} space[20]. As $p_{n,0}^k$ increases by ϵ, each boundary $\Gamma_n^{kl}, l = 1 \ldots K$ of the sets D_n^k and D_n^l shifts away from the set D_n^k in a parallel fashion (some boundaries Γ_n^{kl} can be empty sets). The distance between the old and the new boundaries of the sets is $\frac{1}{|a_n^k - a_n^l|}\epsilon$. Therefore, if I differentiate function $\int_{D_n^k(p_{n,0})} h(x_n) f_n(x_n) dx_n$ (note that the set of integration is $D_n^k(p_{n,0})$) I obtain

$$\frac{\partial}{\partial p_{n,0}^k} \int_{D_n^k(p_{n,0})} h(x_n) f_n(x_n) dx_n = \sum_{i \neq k} \frac{1}{|a_n^k - a_n^i|} \int_{\Gamma_n^{ki}} h(x_n) f_n(x_n)$$

for any continuous function $h(x_n)$. If, on the other hand, I differentiate function $\int_{D_n^l(p_{n,0})} h(x_n) f_n(x_n) dx_n$ (note that the set of integration is $D_n^l(p_{n,0})$) I obtain

$$\frac{\partial}{\partial p_{n,0}^k} \int_{D_n^l(p_{n,0})} h(x_n) f_n(x_n) dx_n = -\frac{1}{|a_n^k - a_n^l|} \int_{\Gamma_n^{kl}} h(x_n) f_n(x_n)$$

I can apply now the above arguments to find the first derivative of $W_n(p_{n,0})$. Note also that for each differentiated term $\int_{D_n^l(p_{n,0})} \left[a_n^l x_n + p_{n,0}^l\right]$ in $W_n(p_{n,0})$ the derivative of the integrand with respect to $p_{n,0}^k$ is one if $l = k$ and zero otherwise. Summing this all up I obtain

[20] For a formal definition of surface integrals see, for example, chapter 14 in "Calculus in Vector Spaces", 1995 by Lawrence Corwin and Robert Szczarba.

$$\frac{\partial W_n}{\partial p_{n,0}^k} = \int_{D_n^k(p_{n,0})} f_n(x_n)dx_n +$$
$$+ \sum_l \frac{1}{|a_n^k - a_n^l|} \int_{\Gamma_n^{kl}} p_n^k(x_n) f_n(x_n) - \sum_l \frac{1}{|a_n^k - a_n^l|} \int_{\Gamma_n^{kl}} p_n^l(x_n) f_n(x_n)$$

By definition, $p_n^k(x_n) = p_n^l(x_n)$ on the boundary of any sets D_n^k and D_n^l so that

$$\frac{\partial W_n}{\partial p_{n,0}^k} = \int_{D_n^k(p_{n,0})} f_n(x_n)dx_n \tag{2.3.24}$$

Existence of the equilibrium follows immediately from equation 2.3.24.

Proposition 2.3.2 *Suppose that Assumption 2.3.1 holds. Suppose also that* $\widehat{p}_0$ *is a solution of the dual optimization problem 2.3.21,* $\overline{D}_n^k(\widehat{p}_{n,0})$ *are the corresponding upper demand sets (defined in 2.3.8), and* $D_n^k(\widehat{p}_{n,0}) = \overline{D}_n^k(\widehat{p}_{n,0})$. *Then the transfers* $\widehat{p}_0$ *and the corresponding demand sets* $D_n^k(\widehat{p}_{n,0})$ *describe the equilibrium of the model.*

The proof follows immediately from the definition of the equilibrium. Condition 2.3.11 of Definition 2.3.1 follows immediately from Assumption 2.3.1. The surplus balance condition 2.3.13 is a constraint in the dual optimization problem 2.3.21 and, therefore, it holds by construction of $\widehat{p}_0$. Finally, mass balance condition 2.3.12 is exactly the first order conditions 2.3.24 of the optimization problem 2.3.21. The result can be generalized as follows.

Theorem 2.3.2 *Equilibrium of* $\mathcal{M}_N^{\mathcal{K}}$ *exists for an arbitrary array of parameters* $\{a_n^k\}$.

Proof[21] is in the Appendix. Next, I find the second derivative of $W_n(p_{n,0})$. Let

[21] Assumption that each distribution of the agents' types has a continuous density function is essential in the proof. It is easy to construct examples with a finite number of agents in each class in which the equilibrium of the model does not exist.

$$\tau_n^{kl} = \frac{1}{|a_n^k - a_n^l|} \int_{\Gamma_n^{kl}} f_n(x_n)\, dx_n \geq 0 \text{ for } k \neq l \tag{2.3.25}$$

Lemma 2.3.1 *The elements $\omega_n^{kl}(p_{n,0})$ of the matrix of the second partial cross derivatives of $W_n(p_{n,0})$ with respect to $p_{n,0}$ are described by the following formula*

$$\omega_n^{kl}(p_{n,0}) = \frac{\partial^2 W_n}{\partial p_{n,0}^k \partial p_{n,0}^l} = \begin{cases} -\tau_n^{kl}, & \text{if } k \neq l \\ \sum_{l \neq k} \tau_n^{kl}, & \text{if } k = l \end{cases} \tag{2.3.26}$$

Proof. As I have shown above $\frac{\partial W_n}{\partial p_{n,0}^k} = \int_{D_n^k(p_{n,0})} f_n(x_n)dx_n$. Taking the partial cross derivative with respect to $p_{n,0}^l$, applying the same argument as I did for the first derivative, and keeping in mind that $h(x_n) \equiv 1$, I obtain

$$\begin{aligned} \frac{\partial^2 W_n}{\partial p_{n,0}^k \partial p_{n,0}^l} &= -\frac{1}{|a_n^k - a_n^l|} \int_{\Gamma_n^{kl}} f_n(x_n) = -\tau_n^{kl} \text{ if } k \neq l \\ \frac{\partial^2 W_n}{\partial p_{n,0}^k \partial p_{n,0}^k} &= \sum_{l \neq k} \frac{1}{|a_n^k - a_n^l|} \int_{\Gamma_n^{kl}} f_n(x_n) = \sum_{l \neq k} \tau_n^{kl}. \end{aligned}$$

■

Let $\Omega_n(p_{n,0}) = (\omega_n^{kl})$. The matrix of the second derivatives $\Omega(p_{n,0})$ has the following properties. The elements on the main diagonal are non-negative, the off-diagonal elements are non-positive, and the sum of the off-diagonal elements in each row is smaller or equal in absolute value than the diagonal element in this row. Any such matrix is positive or semi-positive definite (see, for example, [5]). Therefore, the function $W_n(p_{n,0})$ is convex.

Proposition 2.3.3 *For each n the function $W_n(p_{n,0})$ is a convex function of $p_{n,0}$.*

2.3.5 Uniqueness of the Equilibrium

In this section, I provide conditions under which the equilibrium of the model $\mathcal{M}_N^{\mathcal{K}}$ is unique. In order to do comparative statics exercises, like I do in section 2.4, I need easily verifiable conditions for the uniqueness of the equilibrium transfers to the agents. If $\widetilde{p}_{n,0}$ and $\widehat{p}_{n,0}$ are two equilibrium transfer arrays of decisions in class n then the function $W_n(p_{n,0})$ is constant for any transfer $p_{n,0}^t = t\widetilde{p}_{n,0} + (1-t)\widehat{p}_{n,0}$, $t \in [0,1]$ and, therefore, the vector $\widehat{p}_{n,0} - \widetilde{p}_{n,0}$ is an eigenvector of the second derivative of $W_n(p_{n,0})$ of the eigenvalue of zero. The uniqueness of the transfers is related closely to the number of eigenvectors that correspond to zero eigenvalue of $\Omega_n(p_{n,0})$ for each n.

Eigenvectors of $\Omega_n(p_{n,0})$ when Eigenvalue is Zero

In order to find the dimension of the space of optimal transfers I look at the graph, associated with the matrix $\Omega_n(p_{n,0})$. The graph describes formally the link between the connectedness of set of types X_n (I introduce connectedness later on page 31 in Definition 2.3.3) and the structure of the matrix $\Omega_n(p_{n,0})$. In order to do, so I start with giving some standard definitions from graph theory much of which is quoted directly from "Graphs & Digraphs", 1996, Mike Henning and Ed Palmer.

A *graph* $\mathcal{G}$ is a finite nonempty set of objects called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of $\mathcal{G}$ called *edges*. The vertex set of $\mathcal{G}$ is denoted as $V(\mathcal{G})$, while the edge set is denoted by $E(\mathcal{G})$. Vertices v^1 and v^2 are *adjacent vertices* if $\mathrm{e} = \mathrm{v}^1\mathrm{v}^2$ is an edge of $\mathcal{G}$. A graph $\mathcal{G}$ with vertex set $V(\mathcal{G}) = \left\{\mathrm{v}^1, \ldots, \mathrm{v}^K\right\}$ and edge set $E(\mathcal{G}) = \{\mathrm{e}^1, \ldots, \mathrm{e}^m\}$ can also be described by means of matrices. An *adjacency matrix* $A(\mathcal{G}) = \left[a^{kl}\right]$ is defined as

$$a^{kl} = \begin{cases} 1, & \text{if } \mathrm{v}^k\mathrm{v}^l \in E(\mathcal{G}) \\ 0, & \text{if } \mathrm{v}^k\mathrm{v}^l \notin E(\mathcal{G}) \end{cases}$$

Two vertices v^k and v^l of the graph $\mathcal{G}$ are connected if there exists a *path* (set of edges) that connects the vertices. Graph $\mathcal{G}$ is *connected* if every two vertices of the graph are connected. The relation 'is connected to' is an equivalence relation on the vertex set of graph $\mathcal{G}$. Each subgraph induced by the vertices in a resulting equivalence class is called a *component* of $\mathcal{G}$.

i apply this graph theoretical definitions in the analysis of the number of eigenvectors that correspond to the eigenvalue of zero of $\Omega_n(p_{n,0})$. Let $A_n(p_{n,0}) = [a_n^{ij}]$ be defined as

$$a_n^{ij} = \begin{cases} 1, & \text{if } \omega_n^{ij} \neq 0 \\ 0, & \text{if } \omega_n^{ij} = 0 \end{cases}$$

$A_n(p_{n,0})$ is an adjacency matrix of some graph. Let $\mathcal{G}_n(p_{n,0})$ be the graph associated with the adjacency matrix $A_n(p_{n,0})$.

Let $\{\mathcal{G}_{n,i}\}_{i=1\ldots\eta}$ be the components of $\mathcal{G}_n(p_{n,0})$. Each component $\mathcal{G}_{n,i}$ corresponds to a subset of columns $\mathcal{I}_{n,i} \subset \mathcal{I}$ in the matrix $\Omega_n(p_{n,0})$, where $\mathcal{I} = \{1, 2, \ldots, K\}$ is the set of all columns. For convenience, I use the same terminology for the columns and decisions as for the graph vertices. Thus, I say that *two columns (decisions) are adjacent in* $\Omega_n(p_{n,0})$ if the corresponding two vertices are adjacent in the graph and *two columns (decisions) are connected in* $\Omega_n(p_{n,0})$ if the corresponding vertices are connected. Note that two columns (decisions) k and l are adjacent if $\omega_n^{kl} \neq 0$ and connected[22] if $k, l \in \mathcal{I}_{n,i}$ for some i.

Consider the following vectors $\upsilon_{n,i} = \left[\upsilon_{n,i}^l\right]^{l=1\ldots K}$ for each n and i

[22]If at transfers $p_{n,0}$ decision k is taken by a zero measure of agents in class n then it is not connected in $\Omega_n(p_{n,0})$ to any other decision. In this case any off-diagonal element $\omega_n^{kl} = 0$ in row k.

$$\upsilon_{n,i}^{l}=\begin{cases}1, & \text{if } l\in\mathcal{I}_{n,i}\\ 0, & \text{if } l\notin\mathcal{I}_{n,i}\end{cases} \tag{2.3.27}$$

Fix n and transfers $p_{n,0}$. Let $\mathcal{E}_n(p_{n,0})$ denote the linear space spanned by the eigenvectors of $\Omega_n(p_{n,0})$ that correspond to the eigenvalue of zero. First, I prove the following lemma.

Lemma 2.3.2 *The vectors $\upsilon_{n,i}$, $i=1\ldots\eta$, defined in 2.3.27, are elements of $\mathcal{E}_n(p_{n,0})$.*

Proof is in the Appendix. The space of $\mathcal{E}_n(p_{n,0})$ is, by definition, the space of solutions of the linear system $\Omega_n(p_{n,0})\upsilon=0$. I show next that the set of vectors $\{\upsilon_{n,i}\}_{i=1\ldots\eta}$ forms a basis of $\mathcal{E}_n(p_{n,0})$.

Lemma 2.3.3 *The vectors $\upsilon_{n,i}$, $i=1\ldots\eta$, defined in 2.3.27, form a basis of the space $\mathcal{E}_n(p_{n,0})$.*

Proof is in the Appendix. The following corollary follows immediately from the proof of lemma 2.3.3.

Corollary 2.3.1 *Let υ be an eigenvector of $\Omega_n(p_{n,0})$ that corresponds to the eigenvalue of zero ($\upsilon\in\mathcal{E}_n(p_{n,0})$). Then $\upsilon^k=\upsilon^l$ for any connected columns k and l of the matrix $\Omega_n(p_{n,0})$.*

Uniqueness

Let

$$\mathcal{K}_n^{+}(p_{n,0})=\left\{k:\mu_n\left(D_n^k\right)>0\text{ , given }p_{n,0}\right\} \tag{2.3.28}$$

be the set of decisions that are taken by a positive measure of agents in set X_n given the vector of transfers $p_{n,0}$. If $p_{n,0}$ is an equilibrium transfer matrix then mass balance condition 2.3.12 implies that the set $\mathcal{K}_n^+(p_{n,0})$ is independent of n. Let

$$\mathcal{K}^+(p_0) = \mathcal{K}_n^+(p_{n,0}) \tag{2.3.29}$$

denote the set of decisions taken by a positive measure of agents in X_n for *any* n.

First, I prove the following lemma.

Lemma 2.3.4 *Suppose that $\widetilde{p}_0$ and $\widehat{p}_0$ are two equilibrium transfer matrices and k and l are two decisions that are connected in $\Omega_n(\widetilde{p}_{n,0})$ at transfers $\widetilde{p}_{n,0}$. Then $\widehat{p}^k_{n,0} - \widetilde{p}^k_{n,0} = \widehat{p}^l_{n,0} - \widetilde{p}^l_{n,0}$.*

Proof is in the Appendix. Next I need the following definition.

Definition 2.3.3 *Suppose that the measure μ has a continuous density function $f(x)$. I say that set X is connected with respect to measure μ if there does not exist a pair of disjoint sets A and B such that $\mu(A) > 0$, $\mu(B) > 0$, and $f(x) = 0$ for any $x \in X \setminus (A \cup B)$.*

Lemma 2.3.5 *Suppose that the measure μ_n has a continuous density function $f_n(x)$. If a set of agents types $Y \in X_n$ is connected with respect to μ_n and decisions k and l are such that $\mu_n(Y \cap D_n^k) > 0$ and $\mu_n(Y \cap D_n^l) > 0$, where demand sets D_n^k and D_n^l are constructed for some transfer vector $p_{n,0}$, then decisions k and l are connected in $\Omega_n(p_{n,0})$ at transfers $p_{n,0}$.*

Proof is in the Appendix. Now I am ready to formulate the main result about the uniqueness of the equilibrium transfers.

Theorem 2.3.3 *Suppose that Assumption 2.3.1 holds, there is a continuum of agents endowed with multi-dimensional types in each class n, and for each*

n the distribution of types μ_n has a continuous density function $f_n(x)$. Then the following statements are true.

(i) The set of partitions, generated by the set of solutions of the dual minimization problem 2.3.21, is a singleton.

(ii) Suppose that for some n, the set of types X_n is connected with respect to μ_n. Also suppose that the transfers of decision k_0 satisfy $\widetilde{p}^{k_0}_{n,0} = \widehat{p}^{k_0}_{n,0}$ for some decision $k_0 \in \mathcal{K}^+(\widehat{p}_0)$. Then $\widetilde{p}^{k}_{n,0} = \widehat{p}^{k}_{n,0}$ for any $k \in \mathcal{K}^+(\widehat{p}_0)$.

(iii) For any class n, the vector of optimal transfers $\widehat{p}_{n,0}$ is determined uniquely if the set of types X_n is connected with respect to μ_n and a positive measure of agents are unmatched in class n.

2.4 Applications

So far I have presented a theory of the general N-lateral, K-decision matching model for which I have provided conditions for existence and uniqueness of equilibrium. In this section, I give two application of the marriage market in which the equilibrium of the model is described analytically.

In the first example, I show how redistribution of income affects total production of public goods in the families.

In the second example, I describe the solution of the model analytically in the case that matches between individuals are determined by their choice of education level and age of marriage. I assume that there is some form of complementarity among the choices. For some parameters, in equilibrium, the types are partitioned into two sets, which can conveniently be interpreted as a set of high types and a set of low types. For these parameters, high-type individuals obtain a high level of education and decide to start their families later in life and low-type individuals obtain a low level of education and decide to start their families earlier in life.[23] The example illustrates how complementarity among choices produces assortative matching in the market.[24] In principle, the analytical description of the equilibrium also allows me to do various comparative static exercises. I provide one such comparative static. I show how a change in the cost of education for females affects transfers as well as the education and age of each partner in a match.

It is important to emphasize that these applications are *illustrative*. The objective of the applications is to show how the theory can be applied to explain *selection* in the market in which different types of individuals produce different types of families. The selection determines the equilibrium matching and family structures.

2.4.1 Redistribution of Income and Production of Public Good in the Families

Let's consider the following simple model. There are two classes, $g \in \{1, 2\}$, of individuals: males and females. A male type, $x_1 \in [0, 1]$, and a female type, $x_2 \in [0, 1]$, is interpreted as the individual's income. The distribution of types is denoted by μ_1 for males and by μ_2 for females. A matched pair chooses a decision which is an element of the set $\mathcal{K} = \{0, 1, 2\}$. Decision $k \in \{1, 2\}$ is interpreted as the quantity of the public good produced in the family. For example, it could be the number of kids produced in the family. The surplus generated by a match between a male of type x_1 and a female

[23] In general, other types of families may also be produced in equilibrium (a female with high level of education may be matched with a male with low level of education, etc). A particular matching structure depends on the parameters of the models. The described matching is, in a sense, a 'representative' matching.

[24] In the example, assortative matching means that a subset of high-type males is matched to a subset of high-type females and a subset of low-type males is matched to a subset of low-type females.

of type x_2 that produces k units of public good, is

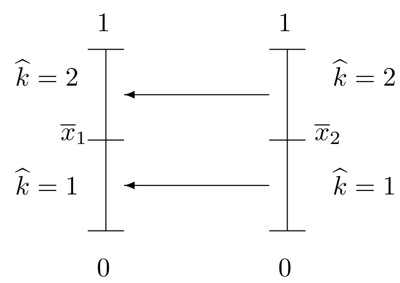
$$u(x_1, x_2, k) = a_1^k x_1 + a_2^k x_2 + a_0^k \tag{2.4.1}$$

I assume that

$$a_1^2 > a_1^1 \qquad a_2^2 > a_2^1 \tag{2.4.2}$$

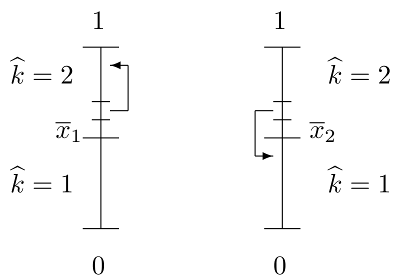
that is, the individuals of higher types have a higher valuation of the public good. For simplicity, I also assume that (1) $a_1^1 = a_2^1 = 0$, (2) $a_0^1 = 0, a_0^2 < 0$ (the restriction implies that while the value of decision $k = 2$ increases faster with the type of an agent than the value of $k = 1$ it also has a higher fixed cost $-a_0^2$.), and (3) $a_0^0 = -\infty$ (the last restriction implies that in equilibrium there are no unmatched individuals).

Let p be the transfer from a male to his matched female if $k = 2$. If $k = 1$ then both individuals in a match obtain zero utility. If two matched individuals choose $k = 2$ then the male obtains utility $u_1 = a_1^2 x_1 - p$ and the female obtains utility $u_2 = a_2^2 x_2 + (a_0^2 + p)$. The equilibrium of the model is described in the figure below.



where $\overline{x}_1 = \frac{\widehat{p}}{a_1^2}$ and $\overline{x}_2 = -\frac{a_0^2 + \widehat{p}}{a_2^2}$ and the equilibrium transfer $\widehat{p}$ is found from the condition $\mu_1([0, \overline{x_1}) = \mu_2([0, \overline{x_2})$. In the figure, the females of types $[0, \overline{x}_2]$ are matched to the males of types $[0, \overline{x}_1]$ in any one-to-one fashion and a matched pair chooses $k = 1$. The females of types $[\overline{x}_2, 1]$ are matched to the males of types $[\overline{x}_1, 1]$ in any one-to-one fashion and a matched pair chooses $k = 2$. This matching function (in which low types are matched to low types and high types are matched to high types) has a simple intuitive explanation. The choice of decision k by an agent depends on the constant part a_0^k and the value of the variable term $a_n^k x_n$ that the agent obtains. Decision $k = 1$ generates a higher constant part and a lower value of the variable part. Therefore, the decision is chosen by low type individuals. The reverse is true for decision $k = 2$.

Next I perform the following comparative static exercise. I change the distribution of types in the following manner. Select any positive mass m of female types $x_2 > \overline{x}_2$ and decrease each of these income types by an amount δ so that income becomes less than $\overline{x}_2$. At the same time, take the positive mass m of males of types $x_1 > \overline{x}_1$ and increase each of these income types by δ. An example of this change in distributions is illustrated in the figure below.



As the distributions of the types change, the measure of the set $[\overline{x}_1, 1]$ does not change while the measure of the set $[\overline{x}_2, 1]$ decreases by m. To bring the system back into the equilibrium I increase the transfer p so that $\overline{x}_1$ increases and $\overline{x}_2$ decreases. In the new equilibrium, the total quantity of the produced public good is smaller than in the original equilibrium so that the distribution of income is not neutral in this model.

Since the distribution is not neutral, we might be interested in fixing the aggregate incomes of all families to be I and asking which distribution of income produces the largest and the smallest quantities of the public good. To answer the question I need to solve the following optimization problem [25]

$$\begin{cases} \max_{\mu_1,\mu_2} \mu_1\left([\overline{x}_1, 1]\right) \\ \int_0^1 x_1 d\mu_1(x_1) + \int_0^1 x_2 d\mu_2(x_2) = I \\ \mu_1\left([\overline{x}_1, 1]\right) = \mu_2\left([\overline{x}_2, 1]\right) \end{cases} \tag{2.4.3}$$

The problem has a simple and intuitively obvious solution. If $a_1^1 > a_2^1$ then $\mu_1\left(\{I\}\right) = 1$ (so that all income is given to the males) and $\mu_2\left(\{0\}\right) = 1$ (so that no income is given to the females). That is, there is only one type of male and one type of female so that the distribution is degenerate. Any arbitrary match is possible and transfers are not uniquely determined. If $a_1^1 < a_2^1$ then the quantity of the public good is maximized if all the income is given to the females.

The opposite rule minimizes the total quantity of the produced public good. If $a_1^1 < a_2^1$ then $\mu_1\left(\{I\}\right) = 1$ and $\mu_1\left(\{0\}\right) = 1$. If $a_1^1 > a_2^1$ then $\mu_1\left(\{I\}\right) = 0$ and $\mu_1\left(\{I\}\right) = 1$.

2.4.2 Marriage Market with Complementary Choices

I consider an example of the marriage market in which each individual chooses a level of education and an age of marriage. The model is set up as a static model[26] . However, I allow the matched partners to choose a different age of marriage in the model. To interpret this type of matches in the static model I illustrate by example that the equilibrium of the static model can be

[25]This problem finds the distributions of types that maximizes the total quantity of public good. To solve the opposite problem I need to change max to min in the problem.

[26]The model is an application of a static $\mathcal{M}_N^{\mathcal{K}}$ model

represented as a *steady state* equilibrium of a multi-period marriage market. Under the multi-period interpretation a new generation enters the market each period and each individual chooses whether to marry in the period of entry (which corresponds to the decision marry early in the static model) or to marry in the subsequent period (which corresponds to the decision to marry late). The matches in which agents choose different age to marry are interpreted as cross-generation marriages.

Now I proceed with a formal description of the static model. Let's consider the following marriage market. Male type[27] is denoted by a scalar $x_1 \in [0, \overline{\vartheta}_1] \subset \mathbb{R}$ and female type is denoted by $x_2 \in [0, \overline{\vartheta}_2] \subset \mathbb{R}$. The distribution of male types is denoted by μ_1 and the distribution of female types is denoted by μ_2. If a male and a female match, they make a decision k which is described by a vector $k = (k_1, k_2, l_1, l_2)$, where $k_1 \in \{0, 1\}$ denotes the male's choice of education, $k_2 \in \{0, 1\}$ denotes the female's choice of education ($k_1 = 0$ and $k_2 = 0$ denote low level of education and $k_1 = 1$ and $k_2 = 1$ denote high level of education) , $l_1 \in \{0, 1\}$ denotes the male's choice of the age of marriage, and $l_2 \in \{0, 1\}$ denotes the female's choice of the age of marriage ($l_1 = 0$ and $l_2 = 0$ denote the choice to marry early and $l_1 = 1$ and $l_2 = 1$ denote the choice to marry late[28]). Given the types x_1 and x_2 of the matched individuals and the decision $k = (k_1, k_2, l_1, l_2)$ made in the match, the surplus generated in the match is

$$u(x_1, x_2, k) = a_1^{k_1} x_1 + a_2^{k_2} x_2 + [b_1^{k_1,l_1} + b_2^{k_2,l_2} + b^{l_1,l_2}] \tag{2.4.4}$$

In this example of the marriage market, I analyze, in detail, the case in which weak complementarity exists between any two choices of an individ-

[27]The type variable may have different interpretations. It could be, for example, ability of an individual, or income saved by the individual's parents for his education.

[28]In the multi-period market $l_1 = 0$ and $l_2 = 0$ denote the choice to marry at the period of entry and $l_1 = 1$ and $l_2 = 1$ denote the choice to marry in the subsequent the period

ual and strict complementarity exists between the choice of education and the type of an individual. I define complementarity similar to how it is defined between types in the matching literature[29]. For example, for any given choices l_1, l_2, k_2 and any given female's type x_2, a male's choice of $k_1 \in \{0, 1\}$ is strictly complementary to the male's type $x_1 \in [0, \overline{\vartheta}_1]$ if for $k_1'' = 1 > 0 = k_1'$ and any $x_1'' > x_1'$

$$u^{(1,k_2,l_1,l_2)}(x_1'', x_2) + u^{(0,k_2,l_1,l_2)}(x_1', x_2) > u^{(1,k_2,l_1,l_2)}(x_1', x_2) + u^{(0,k_2,l_1,l_2)}(x_1'', x_2) \tag{2.4.6}$$

Using 2.4.4, it is immediate that condition 2.4.6 is equivalent to

$$a_1^1 > a_1^0 \tag{2.4.7}$$

Similarly, a female's choice of education is strictly complementary to the female's type x_2 if

$$a_2^1 > a_2^0 \tag{2.4.8}$$

Analogously, I define weak complementarity between any two choices of an individual. For example, for any given types x_1 and x_2 and any given choices l_2, k_2 a male's choice k_1 is weakly complementary to the male's choice l_1 if

$$u^{(1,k_2,1,l_2)}(x_1, x_2) + u^{(0,k_2,0,l_2)}(x_1, x_2) \geq u^{(1,k_2,0,l_2)}(x_1, x_2) + u^{(0,k_2,1,l_2)}(x_1, x_2)$$

or, equivalently,

$$b_1^{1,1} + b_1^{0,0} \geq b_1^{1,0} + b_1^{0,1} \tag{2.4.9}$$

[29]In the matching literature the surplus function $u(x_1, x_2)$ typically depends only on types of the individuals and the types are assumed to be complements. Formally, complementarity between the types is defined as the following condition on the surplus function. For any $x_1'' > x_1'$ and $x_2'' > x_2'$

$$u(x_1'', x_2'') + u(x_1', x_2') \geq u(x_1'', x_2') + u(x_1', x_2'') \tag{2.4.5}$$

Similarly, a female's choices k_2 is weakly complementary to the female's choice l_2 if

$$b_2^{1,1} + b_2^{0,0} \geq b_2^{1,0} + b_2^{0,1} \tag{2.4.10}$$

and a male's choices l_1 is weakly complementary to a female's choice l_2 if

$$b^{1,1} + b^{0,0} \geq b^{1,0} + b^{0,1} \tag{2.4.11}$$

I also assume that

$$a_1^{k_1} \geq 0 \qquad a_2^{k_2} \geq 0 \tag{2.4.12}$$

that is, individuals of higher types generate higher surplus.

Before I describe the equilibrium analytically, I need to define it. Note that in Definition 2.3.1 of the equilibrium I consider a market for each possible decision of a matched pair. In equilibrium, the transfers between married partners clear each of these markets. In the example of this section, I redefine the equilibrium in such a way that the number of markets is reduced to a single market[30]. For a surplus function of the form 2.4.4 I consider only the market for a female's choice to marry late [31]. Thus, I need to construct only the transfer p that a male pays to a female who chooses to marry late. An equilibrium in the marriage market is described by (1) the market clearing transfer, (2) the corresponding matches, and (3) the optimal choices of different types of individuals. In section 2.6.6, I illustrate by example that the equilibrium constructed in this example can be equivalently represented as the equilibrium introduced in Definition 2.3.1. Therefore, the definition of the equilibrium that I use in this section simplifies the analysis but does not introduce a new equilibrium concept of the model.

[30] I do not have a theory yet that describes how in general to reduce the number of markets, that correspond to different decisions, for a given arbitrary form of the surplus function. Analysis of the problem is a part of my current research.

[31] Alternatively, I may consider the market for male's choice to marry late or the market that corresponds to some other choice.

Let's now consider the market for a female's choice $l_2 = 1$. First, I define the demand sets and the demand correspondences for males and females. For a given transfer p, the *upper* demand set $D_1(p)$ for males is the set of types of males who weakly prefer to be matched with a female who chooses $l_2 = 1$. That is,

$$\overline{D}_1(p) = \left\{ x_1 : \max_{(k_1,l_1)} \left(a_1^{k_1} x_1 + b_1^{k_1,l_1} + b^{l_1,1} \right) - p \geq \max_{(k_1,l_1)} \left(a_1^{k_1} x_1 + b_1^{k_1,l_1} + b^{l_1,0} \right) \right\} \tag{2.4.13}$$

Similarly, the *upper* demand set for females is the set of types of females who weakly prefer $l_2 = 1$ over $l_2 = 0$.

$$\overline{D}_2(p) = \left\{ x_2 : \max_{k_2} \left(a_2^{k_2} x_2 + b_2^{k_2,1} \right) + p \geq \max_{k_2} \left(a_2^{k_2} x_2 + b_2^{k_2,0} \right) \right\} \tag{2.4.14}$$

The *lower* demand set for males $\underline{D}_1(p)$ and the *lower* demand set for females $\underline{D}_2(p)$ is defined analogously using strict inequalities in (2.4.13) and (2.4.14). A demand set for males is any set $D_1(p)$ such that

$$\underline{D}_1(p) \subseteq D_1(p) \subseteq \overline{D}_1(p) \tag{2.4.15}$$

Similarly, a demand set for females is any set $D_2(p)$ such that

$$\underline{D}_2(p) \subseteq D_2(p) \subseteq \overline{D}_2(p) \tag{2.4.16}$$

To define the equilibrium, I use the concept of demand correspondences[32]. The demand correspondence for males $d_1(p)$ is defined as

$$d_1(p) = \left[\mu_1 \left(\underline{D}_1(p) \right), \mu_1 \left(\overline{D}_1(p) \right) \right] \tag{2.4.17}$$

and the demand correspondence for females $d_2(p)$ is defined as

$$d_2(p) = \left[\mu_2 \left(\underline{D}_2(p) \right), \mu_2 \left(\overline{D}_2(p) \right) \right] \tag{2.4.18}$$

Now, I define equilibrium in the market as follows.

---

[32] For any transfer p the demand correspondence for males is the interval of measures of male demand sets $D_1(p)$ that satisfy 2.4.15.

Definition 2.4.1 *Equilibrium of the marriage market is defined as*

1. *a transfer $\widehat{p}$ from a male to his female partner*[33] *if she chooses $l_2 = 1$,*

2. *the male and female demand sets $D_1(\widehat{p})$ and $D_2(\widehat{p})$ that satisfy (2.4.15) and (2.4.16),*

3. *the optimal choices $\widehat{k}_1(x_1), \widehat{l}_1(x_1)$, and $\widehat{l}_2^m(x_1)$ of males of types $x_1 \in [0, \overline{\vartheta}_1]$, and the optimal choices $\widehat{k}_2(x_2)$ and $\widehat{l}_2(x_2)$ of females of types $x_2 \in [0, \overline{\vartheta}_2]$*

such that the demand sets $D_1(\widehat{p})$ and $D_2(\widehat{p})$ have a common measure.

$$\mu_1(D_1(\widehat{p})) = \mu_2(D_2(\widehat{p})) \tag{2.4.19}$$

Note that (2.4.19) is possible only if

$$d_1(\widehat{p}) \cap d_2(\widehat{p}) \neq \emptyset \tag{2.4.20}$$

The demand sets $D_1(\widehat{p})$ and $D_2(\widehat{p})$ generate the equilibrium matching function. In the equilibrium males of type $x_1 \in D_1(\widehat{p})$ are matched in any one-to-one fashion to females of type $x_2 \in D_2(\widehat{p})$. Similarly, males of type $x_1 \notin D_1(\widehat{p})$ are matched in any one-to-one fashion to females of type $x_2 \notin D_2(\widehat{p})$.

Now that the equilibrium and each of its components have been defined, I am able to give a step by step process that describes informally the construction of the equilibrium. First, I construct demand correspondences $d_1(p)$ and $d_2(p)$ for $p \in (-\infty, \infty)$. Given $d_1(p)$ and $d_2(p)$, I find a transfer $\widehat{p}$ and construct the sets $D_1(\widehat{p})$ and $D_2(\widehat{p})$ such that (2.4.20) holds. Given $\widehat{p}$, I find the optimal choices $\widehat{k}_1(x_1), \widehat{l}_1(x_1), \widehat{l}_2^m(x_1)$ of males of types $x_1 \in [0, \overline{\vartheta}_1]$, and the optimal choices $\widehat{k}_2(x_2)$ and $\widehat{l}_2(x_2)$ of females of types $x_2 \in [0, \overline{\vartheta}_2]$.

In the next section, I provide details of a general construction of equilibrium in the case of complementary choices.

[33] The transfer can be either positive or negative.

Complementary Choices

In this section I assume that parameters of the surplus function 2.4.4 satisfy the properties $2.4.7 - 2.4.12$.

In order to derive the demand correspondences, I need to derive the upper and lower demand sets. In order to do so, I need to construct the optimal choices of the females of different types. I begin by deriving a female's optimal choice of education k_2, given her type x_2 and choice l_2. If a female of type x_2 chooses $l_2 = 0$, then the choice $k_2 = 0$ gives her a higher utility than the choice $k_2 = 1$ if

$$a_2^0 x_2 + b_2^{0,0} \geq a_2^1 x_2 + b_2^{1,0} \tag{2.4.21}$$

Similarly, for $l_2 = 1$, the optimal choice is $k_2 = 0$ if

$$a_2^0 x_2 + b_2^{0,1} \geq a_2^1 x_2 + b_2^{1,1} \tag{2.4.22}$$

The optimal choice of k_2 conditional on l_2^f and x_2 (denoted as $\widetilde{k}_2(l_2^f, x_2)$) can be derived from 2.4.21 and 2.4.22. In order to do so, let[34] $x_2^l = \frac{b_2^{0,1} - b_2^{1,1}}{a_2^1 - a_2^0}$ and $x_2^r = \frac{b_2^{0,0} - b_2^{1,0}}{a_2^1 - a_2^0}$. The function $\widetilde{k}_2(l_2^f, x_2)$ is illustrated in Figure 2.4.1. Whenever a female whose type is less than $x_2^r = \frac{b_2^{0,0} - b_2^{1,0}}{a_2^1 - a_2^0}$ chooses to marry early, the female also chooses low education. Whenever a female whose type is less than $x_2^l = \frac{b_2^{0,1} - b_2^{1,1}}{a_2^1 - a_2^0}$ chooses to marry late, the female also chooses low education.

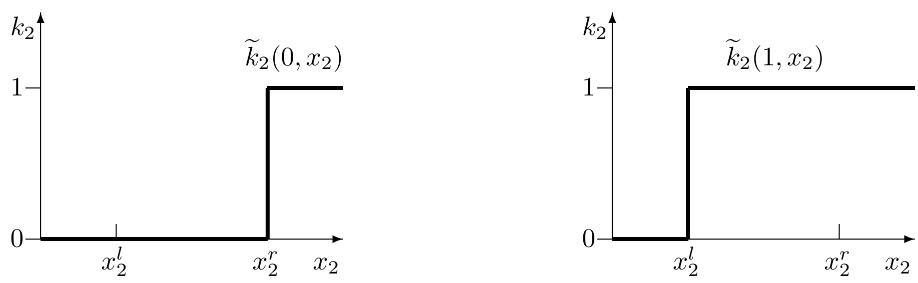


Figure 2.4.1 *A female's optimal choice $\widehat{k}_2$, given her choice l_2 and her type x_2.*

[34]Note, that from 2.4.8 and 2.4.10 it follows that $x_2^r \geq x_1^l$

Now I can find the optimal choices $\widehat{k}_2$ and $\widehat{l}_2$ of a female of type x_2. If $x_2 < x_2^l$ and $l_2 = 0$ then $\widehat{k}_2 = 0$ and her utility is $u_2 = a_2^0 x_2 + b_2^{0,0}$. If $x_2 < x_2^l$ and $l_2 = 1$ then $\widehat{k}_2 = 0$ and her utility is $u_2 = a_2^0 x_2 + b_2^{0,1} + p$. Therefore, the female of type $x_2 < x_2^l$ chooses $l_2 = 1$ whenever $p \geq b_2^{0,0} - b_2^{1,0}$. Similarly, if $x_2 \in [x_2^l, x_2^r]$ then the female chooses $l_2 = 1$ whenever $p \geq (b_2^{1,0} - b_2^{0,1}) - (a_2^1 - a_2^0)x_2$, and, if $x_2 \geq x_2^r$, then the female chooses $l_2 = 1$ whenever $p \geq b_2^{1,0} - b_2^{1,1}$. For any transfer p, Figure 2.4.2 illustrates the set of females' types x_2 who choose $l_2 = 1$ as well as the upper and lower demand sets for females $\overline{D}_2(p)$ and $\underline{D}_2(p)$.

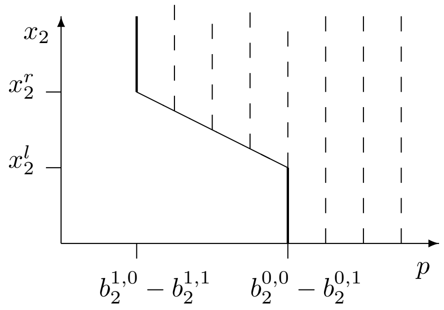


Figure 2.4.2 *The dashed region and its boundary show the set of pairs (p, x_2) such that, given transfer p, the type x_2 female weakly prefers $l_2 = 1$ over $l_2 = 0$. The upper demand set $\overline{D}_2(p)$ is the set of types x_2 for which (p, x_2) belongs to either the interior or the boundary of the dashed region and the lower demand set $\underline{D}_2(p)$ is the set of types x_2 for which (p, x_2) belongs to the interior of the dashed region.*

For each p, the measures of the lower and upper demand sets for females determine the demand correspondence for females $d_2(p)$. The demand correspondence for females $d_2(p)$ is shown in Figure 2.4.3.

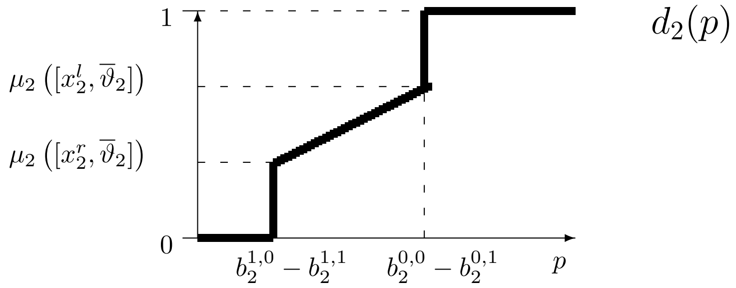


Figure 2.4.3 *Demand correspondence for females*

In a similar fashion I derive the demand correspondence for males $d_1(p)$. If I let[35]

$$\widetilde{b}_1^{k_1, l_2^m} = \max_{l_1} \left[b_1^{k_1, l_1} + b^{l_1, l_2^m} \right] \tag{2.4.24}$$

and note that the transfer p is subtracted from the utility of a male, then the utility function of a male is analogous to the utility function of a female used in the previous construction. Therefore, the optimal choices, the demand sets, and the demand correspondence for males can be derived analogously to how it has been done for females. The males' optimal choice $\widetilde{k}_1(l_2^m, x_1)$ conditional on choice l_2^m and type x_1 is illustrated in Figure 2.4.4.

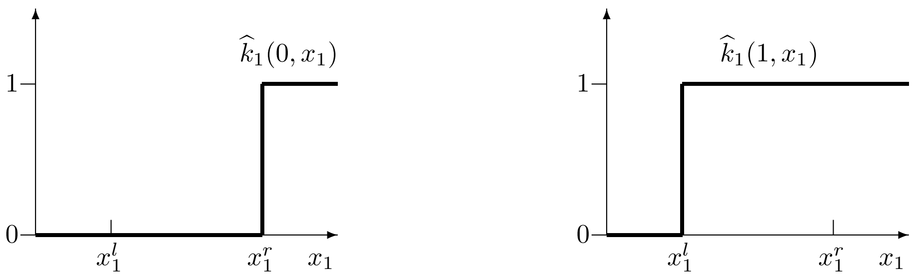


Figure 2.4.4 *A male's optimal choice $\widehat{k}_1$, given his choice l_2^m and his type x_1.*

[35] Note, that from 2.4.9 and 2.4.11 it follows that

$$\widetilde{b}_1^{1,1} + \widetilde{b}_1^{0,0} \geq \widetilde{b}_1^{1,0} + \widetilde{b}_1^{0,1} \tag{2.4.23}$$

In the picture, whenever a male whose types is less than $x_1^r = \frac{\widetilde{b}_1^{0,0} - \widetilde{b}_1^{1,0}}{a_1^1 - a_1^0}$ chooses partner who marries early, the male also chooses low education. Whenever a male whose type is less than $x_1^l = \frac{\widetilde{b}_1^{0,1} - \widetilde{b}_1^{1,1}}{a_1^1 - a_1^0}$ chooses a partner who marries late, the male also chooses low education. (Note, that from 2.4.7 and 2.4.23 it follows that $x_1^r \geq x_1^l$). The upper and lower demand sets $\overline{D}_1(p)$ and $\underline{D}_1(p)$ for males are illustrated in Figure 2.4.5.

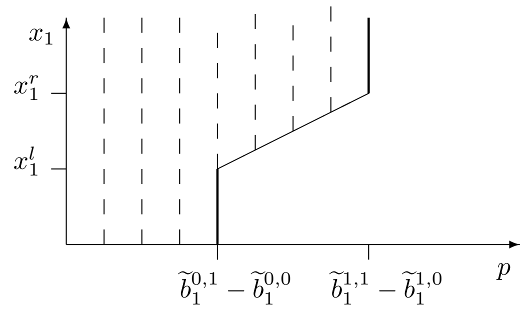


Figure 2.4.5 *The dashed region and its boundary show the set of pairs (p, x_1) such that, given transfer p, the type x_1 male weakly prefers to be matched with a female who chooses $l_2 = 1$. The upper demand set $\overline{D}_1(p)$ is the set of types x_1 for which (p, x_1) belongs either to the interior or the boundary of the dashed region and the lower demand set $\underline{D}_1(p)$ is the set of types x_1 for which (p, x_1) belongs to the interior of the dashed region.*

Using the upper and lower demand sets $\overline{D}_1$ and $\underline{D}_1(p)$ I derive the demand correspondence $d_1(p)$ which is illustrated in Figure 2.4.6.

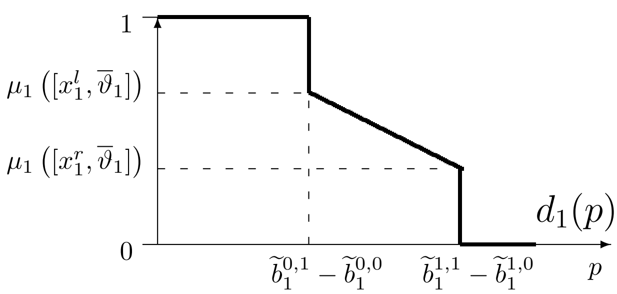


Figure 2.4.6 *Demand correspondence for females*

Now I construct the equilibrium of the model. The equilibrium transfer, optimal decisions, and corresponding matching can be found by following the steps below.

1. Find the transfer $\widehat{p}$ at which the demand correspondences intersect. The transfer $\widehat{p}$ is the equilibrium transfer and $\widehat{m} \in d_1(\widehat{p}) \cap d_2(\widehat{p})$ is the measure of couples who choose $l_2 = 1$ when matched.

2. Using $\widehat{p}$, $\widehat{m}$, and Figures 2.4.2 and 2.4.5, derive which types of females and males choose $l_2 = 1$ and $l_2^m = 1$ and derive the demand sets $D_1(\widehat{p})$ and $D_2(\widehat{p})$.

3. Using $\widehat{l}_2$, $\widehat{l_2^m}$, and Figures 2.4.1 and 2.4.4, derive the females' and males' optimal choices $\widehat{k}_2$ and $\widehat{k}_1$.

4. Finally, using $\widehat{k}_1$, $\widehat{l_2^m}$ derive the males' optimal choice $\widehat{l}_1$ from the following equation

$$\widehat{l}_1 = \arg\max_{l_1} \left[b_1^{\widehat{k}_1, l_1} + b^{l_1, \widehat{l}_m^2} \right] \tag{2.4.25}$$

The qualitative nature of the equilibrium depends on the form of the intersection of the demand correspondences $d_1(p)$ and $d_2(p)$ and on the solution of 2.4.25. I describe two "interesting" cases. In each case, the set of types of males and the set of types of females are partitioned into two subsets: a subset of low-type individuals and a subset of high-type individuals. The low-type individuals choose low level of education while the high-type individuals choose high level of education. In the first case, the matched partners have the same level of education and choose the same age to marry. In the second case, partners may have different levels of education and have different age. Note, that in the second case the equilibrium has a natural interpretation as a steady state in the multi-period setting.

Generally, there are several forms that the intersection of the demand correspondences may take. I provide the context for two cases. The first case occur when the demand correspondences intersect as shown in Figure 2.4.7 and[36]

$$\begin{cases} b_1^{0,0} + b^{0,0} \geq b_1^{0,1} + b^{1,0} \\ b_1^{1,1} + b^{1,1} \geq b_1^{1,0} + b^{1,0} \end{cases} \tag{2.4.26}$$

.

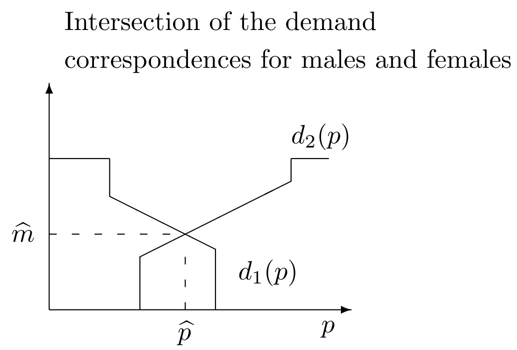


Figure 2.4.7 *This picture illustrates the case for which the equilibrium transfer $\widehat{p}$ belongs to the interval $(b_2^{1,0} - b_2^{1,1}, b_2^{0,0} - b_2^{0,1}) \cap (\widetilde{b}_1^{0,1} - \widetilde{b}_1^{0,0}, \widetilde{b}_1^{1,1} - \widetilde{b}_1^{1,0})$ in which the demand correspondence for females is strictly increasing and the demand correspondence for males is strictly decreasing.*

To describe the equilibrium I follow the four steps outlined above. From the intersection of demand correspondences I find the transfer $\widehat{p}$ and the measure of couples $\widehat{m}$ in which the female marries late. Using Figure 2.4.3, I find that $\widehat{p} \in (b_2^{1,0} - b_2^{1,1}, b_2^{0,0} - b_2^{0,1})$. Using Figure 2.4.2, I find that females of types $[0, \overline{x_2}]$ choose $\widehat{l_2} = 0$ and females of types $[\overline{x_2}, \overline{\vartheta}_2]$ choose $\widehat{l_2} = 1$, where $\overline{x_2} \in (x_2^l, x_2^r)$ and $\mu_2([\overline{x_2}, \overline{\vartheta}_2]) = \widehat{m}$. Using the left panel of Figure 2.4.1, I find that the optimal choice of education of a female of type $x_2 \in [0, \overline{x_2}]$ is $\widehat{k}_2 = 0$ and, using the right panel of Figure 2.4.1, I find that the optimal choice of education of a female of type $x_2 \in [\overline{x_2}, \overline{\vartheta}_2]$is $\widehat{k}_2 = 1$. Analogously, I construct

[36]This condition determines the males' optimal choice $\widehat{l_1}$

the male optimal choices of l_2^m and k_2. The male optimal choice of l_1 is found from equation 2.4.25

The equilibrium is illustrated in Figure 2.4.8.

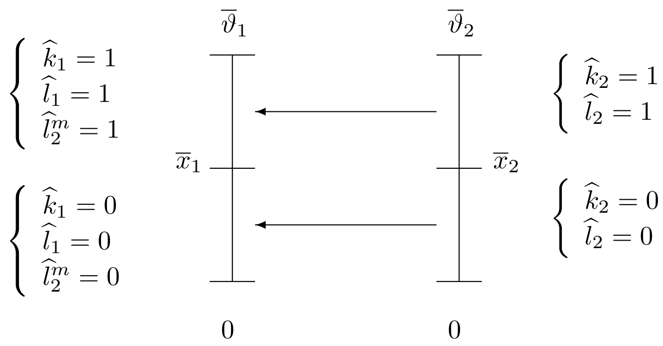


Figure 2.4.8 *In the pictures $\overline{x}_1$ and $\overline{x}_2$ are such that $\mu_1([0, \overline{x}_1]) = \mu_2([0, \overline{x}_2]) = \widehat{m}$. Arrows indicate matched sets and choices are indicated beside each set. Low-type males, $x_1 \in [0, \overline{x}_1]$, are matched in any one-to-one fashion to low-type females $x_2 \in [0, \overline{x}_2]$ and high-type males, $x_1 \in [\overline{x}_1, \overline{\vartheta}_1]$, are matched in any one-to-one fashion to high-type females $x_2 \in [\overline{x}_2, \overline{\vartheta}_2]$.*

The second case under consideration occurs when the demand correspondences intersect as illustrated in the following picture

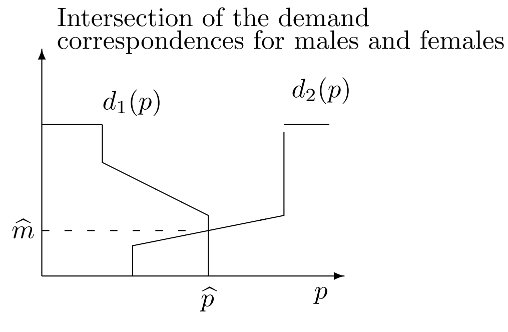


Figure 2.4.9 *This picture illustrates the case for which the equilibrium transfer is $\widehat{p} = \widetilde{b}_2^{1,1} - \widetilde{b}_2^{1,0}$. The demand correspondence for females is strictly increasing at $\widehat{p}$ and the demand correspondence for males is vertical at $\widehat{p}$.*

In this case, following the four steps outlined above leads to the equilibrium illustrated in Figure 2.4.10 when $\left\{ \begin{array}{l} b_1^{0,0} + b^{0,0} \geq b_1^{0,1} + b^{1,0} \\ b_1^{1,1} + b^{1,1} \geq b_1^{1,0} + b^{1,0} \end{array} \right.$.

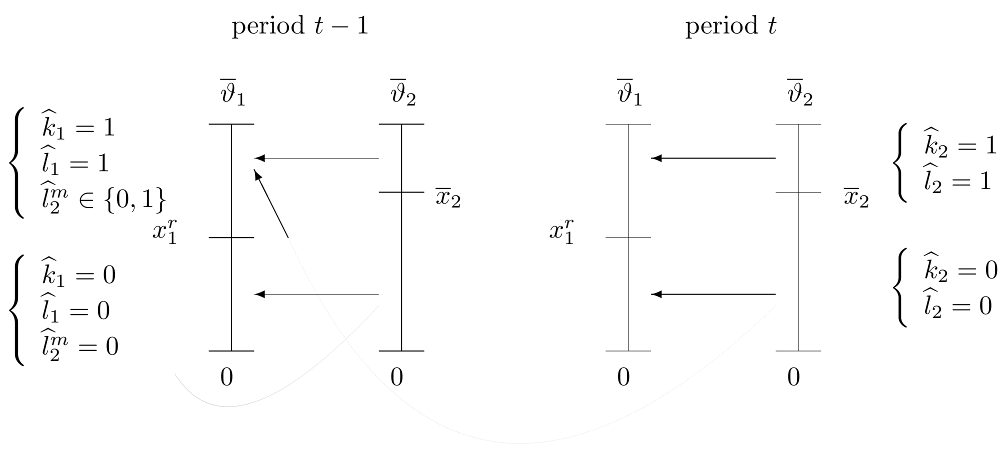


Figure 2.4.10 *The agents of types shown on the left side of the picture enter the marriage market in period $t-1$ and the agents of types shown on the right side of the picture enter the marriage market in period t. In the picture, $\overline{x}_2$ is such that $\mu_2([0,\overline{x}_2]) = \widehat{m}$. Arrows indicate matched sets and choices are indicated beside each set. In the example, some of the high-type males who obtain high education and choose to marry late are matched to low-type females who obtain low education and choose to marry early. In this type of marriage males enter the marriage market in period $t-1$ and wait until period t to match with a female partner.*

Comparative Statics

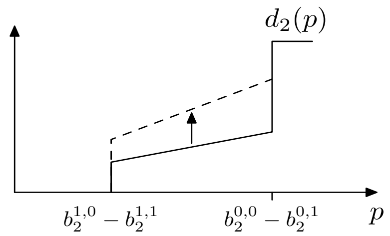
I now use the analytical description of the equilibrium to show how a change in the parameters of the marriage market model affects the equilibrium. I provide only one example that shows how a change in the females' value of the high level of education, a_2^1, affects (i) the choices of education and age to marry and (ii) transfers in the matched couples. I describe how the equilibrium changes as a_2^1 increases. A small difference between the coefficients a_2^1

and a_2^0 can be interpreted, for example, as existence of barriers for females to obtain high-level job positions. The difference $a_2^1 - a_2^0$ can be measured as the effect of females' education on their wages.

¿From the construction of the equilibrium we can see that a change in a_2^1 effects only the cut-offs

$$x_2^l = \frac{b_2^{0,1} - b_2^{1,1}}{a_2^1 - a_2^0} \text{ and } x_2^r = \frac{b_2^{0,0} - b_2^{1,0}}{a_2^1 - a_2^0}$$

for females constructed on page 42. As a_2^1 increases the female demand correspondence shifts as illustrated below.



If a_2^1 is close to a_2^0, the intersection of demand correspondences is as shown in Figure 2.4.9. Figure 2.4.10 illustrates the equilibrium in the market in this case. In this case, the cutoff that separates females by age of marriage (denoted by $\overline{x_2}$) is close to or equal to $\overline{\vartheta_2}$. Therefore, the measure of the set of females of types $x_2 \geq \overline{x_2}$ who marry late and obtain a high level of education is close or equal to zero. The set of males of types $x_1 \geq x_1^r$ who marry late and obtain a high level of education does not change with a change in a_2^1 as long as the demand correspondences intersect as shown in Figure 2.4.9. The males in the set are indifferent about the age of their partner. Some of them are matched with females from their own generation and others are matched with females from a younger generation.

As the value of education for females a_2^1 increases, the cutoff that separates females by age of marriage decreases so that the measure of the set of females who marry late and obtain a high level of education increases, that is, $\overline{x_2}$ decreases as a_2^1 increases. If, after an increase in a_2^1, the demand correspondences still intersect as shown in Figure 2.4.9, then the transfers do not change even though some males who formerly matched with females who marry early now match with females who marry late. This is because the males in any new matches that take place are indifferent among females who marry late or marry early so that these males do not obtain a higher transfer for switching from one partner to the other.

As a_2^1 increases further, the intersection of demand correspondences becomes as shown in Figure 2.4.7. The new equilibrium is shown in Figure 2.4.8. There are no cross-generation marriages in the new equilibrium and there is a complete segregation with respect to education level of the partners.

Similarly, I can analyze the effect of changes in other coefficients of the surplus function (which can be interpreted as education costs, costs of obtaining a job, costs of cross-generation marriages, etc.) on the matching pattern and equilibrium transfers within different types of families.

Non Complementary Choices

Note that matching illustrated in Figures 2.4.8 and 2.4.10 can be naturally interpreted as assortative matching. In each figure, the set of low type males is matched to the set of low type females and the set of high type males is matched to the set of high type females. This result may not be true if the choices of the individuals are not complementary to each other. Let's illustrate this by example. I assume now that all inequalities 2.4.7 - 2.4.12 hold except the inequality 2.4.11 so that choices l_1 and l_2 are not complementary. Moreover, I assume that the parameters of the surplus function are such that

$$\widetilde{b}_1^{0,0} + \widetilde{b}_1^{1,1} < \widetilde{b}_1^{1,0} + \widetilde{b}_1^{0,1} \tag{2.4.27}$$

where $\widetilde{b}_1$ is defined in 2.4.24. The equilibrium can be derived in a similar fashion as it is done in the previous example. In particular, it can be shown that the lower and upper demand sets for males and females are such as shown in Figure 2.4.11.

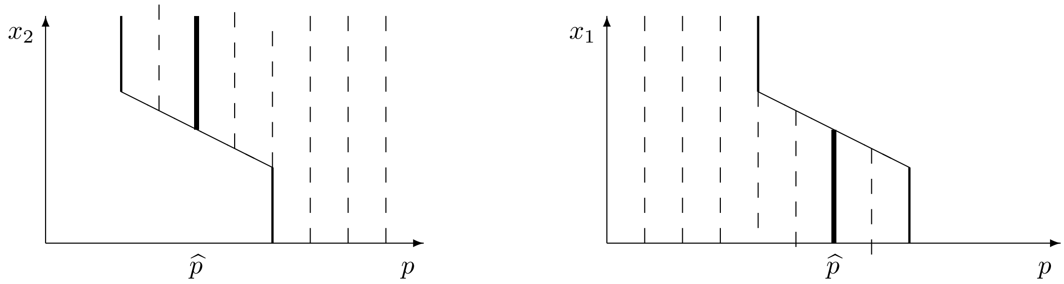
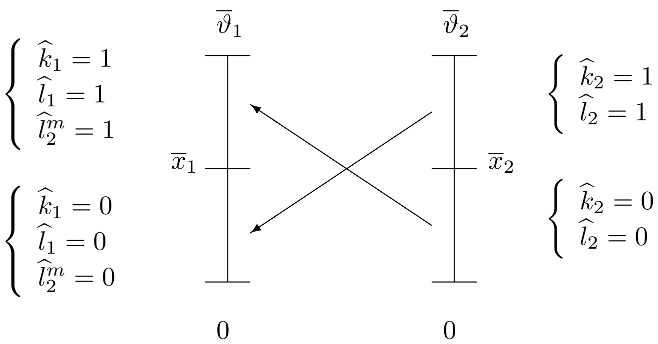


Figure 2.4.11 *The picture illustrates the demand sets for males and females for different transfers p. The bold lines illustrate how the sets are matched to each other at equilibrium transfer $\widehat{p}$.*

If the demand correspondence for males is strictly decreasing at the equilibrium transfer $\widehat{p}$ and the demand correspondence for females is strictly increasing at the equilibrium transfer $\widehat{p}$ then, under certain conditions on $\widetilde{b}$, the equilibrium matching is illustrated by the following figure.



2.5 Conclusion and Extensions

The chapter studies a one-to-...-one N-lateral matching model in which an agent from a matched group of N people cares about a decision made by the group and not about the types of the partners. The chapter focuses on the case in which there is a continuum of agents in each of the N classes, each agent is endowed with a multi-dimensional type, and the distribution of types has a continuous density function. To construct an equilibrium of the model I solve an associated minimization problem. It is shown that the problem is convex and the solution of the problem is the equilibrium of the model. Standard Gauss-Seidel methods can be applied to construct the equilibrium numerically.

The chapter provides easily verifiable conditions for the uniqueness of the equilibrium.

Even in the case of bilateral matching, my model introduces an alternative representation of a standard matching model with no frictions in which utility is transferrable. In the new representation of the bilateral matching model an individual looks for a partner to undertake some project k. The choice of the project and the partner depends on the transfers between the matched individuals as they decide which project to choose, and does not depend directly on the type of the partner. In this chapter, I derive the properties of the equilibrium of the model and show how the equilibrium can be constructed.

To illustrate the contributions of the model in the bilateral setting, I discuss in this chapter two applications of the model. In the first application, I show how the total quantity of public good produced by families depends on the distribution of income between males and females. The second application illustrates how complementarity between the individual choices affects the matching pattern and shows, for example, how a change in the cost of education for females affects the education and age of each partner in a match.

There is a large variety of extensions of the model, some of which are a part of my current research. I mention only two.

If you use the model to interpret data on married couples you would naturally assume that there is some random component that affects the choices of the individuals. There is a natural way to introduce noise in the model. It can be shown that, in the modified model, the equilibrium has analogous properties as in the deterministic model and therefore similar tools show how the equilibrium can be constructed. Alternatively, you can use the deterministic model of this chapter to describe data but you may have to assume that some of the types of agents are not observable.

These and other extensions and applications of the model are a part of my current research.

2.6 Appendix

2.6.1 Equilibrium and Stable Matching

Proposition 2.3.1

(i) Equilibrium matching and transfers generate a stable matching.

(ii) Any stable matching is generated by some equilibrium matching and transfers.

Proof.

(i) Suppose that an array of transfers $p^k_{n,0}$ and corresponding demand sets describe an equilibrium of the model. Then, by definition of the demand set, an agent in class n of type x_n obtains a transfer

$$p_n(x_n) = \max_{l=0}^{N} \left[a^l_n x_n + p^l_{n,0}\right] \tag{2.6.1}$$

Equation 2.6.1 implies that for any decision k

$$\sum_n p_n(x_n) \geq \sum_n \left[a_n^k x_n + p_{n,0}^k\right] = \sum_n a_n^k x_n + a_0^k \tag{2.6.2}$$

and, therefore, condition 2.3.14 of Definition 2.3.2 holds. On the other hand, if agents of types $(x_1, m_2(x_1), \ldots, m_N(x_1))$ are matched into a group then, by definition of the equilibrium matching, they all choose the same decision k and this decision maximizes their utility. That is

$$p_n(m_n(x_1)) = a_n^k m_n(x_1) + p_{n,0}^k \tag{2.6.3}$$

Therefore, condition 2.3.15 of Definition 2.3.2 holds.

(ii) Suppose that functions $(m_2(\cdot), \ldots, m_N(\cdot))$ and corresponding transfers $p_n(x_n)$, $n = 1, \ldots, N$ describe a stable matching. For each n and k let X_n^k denote the set of types of agents in class n that belong to a matched group of types for whom decision k maximizes the match surplus. First, I show that, for any n, k and $x_n \in X_n^k$, the transfer function $p_n(x_n)$ is

$$p_n(x_n) = a_n^k x_n + p_{n,0}^k \tag{2.6.4}$$

where $(p_{n,0}^k)$ is some array of parameters. Suppose that, for some n, k, and $x_n \in X_n^k$, 2.6.4 does not hold, or, equivalently, function $p_n(x_n) - a_n^k x_n$ is not constant on X_n^k. Then there exist two points $\overline{x}_n$ and $\widehat{x}_n$ such that

$$p_n(\widehat{x}_n) - a_n^k \widehat{x}_n < p_n(\overline{x}_n) - a_n^k \overline{x}_n \tag{2.6.5}$$

Suppose that type $\overline{x}_n$ agent is matched with the agents of types $(\overline{x}_1, \ldots, \overline{x}_N)$. By Definition 2.3.2 of stable matching (equation 2.3.14), the sum of transfers of the agents is equal to the surplus, generated by the agents.

$$\sum_{m \neq n} p_m(\overline{x}_m) + p_n(\overline{x}_n) = \sum_{m \neq n} a_m^k \overline{x}_m + a_n^k \overline{x}_n + a_0^k \tag{2.6.6}$$

From 2.6.5 and 2.6.6 it follows that

$$\sum_{m \neq n} p_m(\overline{x}_m) + p_n(\widehat{x}_n) < \sum_{m \neq n} a_m^k \overline{x}_m + a_n^k \widehat{x}_n + a_0^k \tag{2.6.7}$$

But the last inequality contradicts condition 2.3.14 in Definition 2.3.2 of the stable matching. To finish the proof I need to show that, for any n, k, l, and any $x_n \in X_n^k$

$$a_n^k x_n + p_{n,0}^k \geq a_n^l x_n + p_{n,0}^l \tag{2.6.8}$$

Suppose that the inequality does not hold for some $n = 1, \dots, N$, $k, l = 1, \dots, K$, and $x_n \in X_n^k$. That is, there exist indices n, k, l and type $\widehat{x}_n \in X_n^k$ such that

$$a_n^k \widehat{x}_n + p_{n,0}^k < a_n^l \widehat{x}_n + p_{n,0}^l \tag{2.6.9}$$

Let's pick some arbitrary vector of types $(\overline{x}_1, \dots, \overline{x}_n, \dots, \overline{x}_N)$ such that decision l maximizes the surplus if the types form a match. Let's consider the vector of types $(\overline{x}_1, \dots, \widehat{x}_n, \dots, \overline{x}_N)$ in which type $\overline{x}_n$ is replaced by $\widehat{x}_n$. If the types $(\overline{x}_1, \dots, \widehat{x}_n, \dots, \overline{x}_N)$ are matched, then the surplus generated in the match is at least

$$u(\overline{x}_1, \dots, \widehat{x}_n, \dots, \overline{x}_N) \geq \sum_{m \neq n} a_m^l \overline{x}_m + a_n^l \widehat{x}_n + a_0^l \tag{2.6.10}$$

On the other hand, using 2.6.9 and 2.6.10 I obtain that the sum of transfers of agents of types $(\overline{x}_1, \dots, \widehat{x}_n, \dots, \overline{x}_N)$ is

$$\begin{aligned} \sum_{m \neq n} \left[a_m^l \overline{x}_m + p_{m,0}^l\right] + a_n^k \widehat{x}_n + p_{n,0}^k &< \sum_{m \neq n} \left[a_m^l \overline{x}_m + p_{m,0}^l\right] + a_n^l \widehat{x}_n + p_{n,0}^l \\ &\leq u(\overline{x}_1, \dots, \widehat{x}_n, \dots, \overline{x}_N) \end{aligned}$$

which contradicts condition 2.3.14 in Definition 2.3.2 of the stable matching.

■

2.6.2 Existence of the Equilibrium

Theorem 2.3.1 The following statements hold.

(i) For any array of transfers p_0 that satisfies the constraints of the problem 2.3.21 and for any partition vector $\mathcal{P}$ that satisfies the constraints of the problem 2.3.18 the value of the dual objective function is greater or equal to the value of the objective function of the optimal partition problem, that is $W(p_0) \geq V(\mathcal{P})$.

(ii) Suppose that assumption 2.3.1 holds and in each class there is a continuum of agents endowed with multi-dimensional types and the distribution of the types has a continuous density function. Then the optimal array of transfers $\widehat{p_0}$ generates the partition $\widehat{\mathcal{P}}$ of the sets X_n into demand sets D_n^k, defined as in 2.3.10, such that $W(\widehat{p_0}) = V(\widehat{\mathcal{P}})$. The partition $\widehat{\mathcal{P}}$ is a solution to 2.3.18. The transfers $\widehat{p_0}$ and the corresponding demand sets D_n^k describe the equilibrium of the model.

(iii) If $W(\widehat{p_0}) > V(\widehat{\mathcal{P}})$, where $\widehat{p_0}$ is a solution of 2.3.21 and $\widehat{\mathcal{P}}$ is a solution of 2.3.18 then the equilibrium of the model does not exist.

Proof.

(i) Let p_0 be an arbitrary array of transfers that satisfies the constraints of the problem 2.3.21 and let $\mathcal{P}$ be an arbitrary set partition vector that satisfies the constraints of the problem 2.3.18. Then, by definition,

$$V(\mathcal{P}) = \sum_k \left[\sum_n \Big(\int_{\Pi_n^k} a_n^k x_n f_n(x_n) dx_n + \int_{\Pi_n^0} r_n(x_n) f_n(x_n) dx_n \Big) + a_0^k \int_{\Pi_1^k} f_1(x_1) dx_1 \right] \quad (2.6.11)$$

By assumption, the transfers $p^k_{n,0}$ satisfy the constraints of 2.3.21, that is $\sum_n p^k_{n,0} = a^k_0$, and mass balance condition holds, $\int_{\Pi^k_1} f_1(x_1)dx_1 = \int_{\Pi^k_n} f_n(x_n)dx_n$ for any n. Therefore, if I substitute out the term $a^k_0 \int_{\Pi^k_1} f_1(x_1)dx_1$ in 2.6.11 and rearrange the sum, I can rewrite $V(\mathcal{P})$ as

$$V(\mathcal{P}) = \sum_n \left[\sum_k \int_{\Pi^k_n} \left(a^k_n x_n + p^k_{n,0}\right) f_n(x_n)dx_n + \int_{\Pi^0_n} \left(a^0_n x_n + r^0_n + p^0_{n,0}\right) f_n(x_n)dx_n \right] \tag{2.6.12}$$

or, using notation 2.3.6 for $p^k_n(x_n)$ and taking summation with respect to k inside the integral I can rewrite it as

$$V(\mathcal{P}) = \sum_{n=1\ldots N} \int_{X_n} \left(\sum_{k=0\ldots K} p^k_n(x_n) \mathbf{1}_{\Pi^k_n}(x_n) \right) f_n(x_n)dx_n \tag{2.6.13}$$

where $\mathbf{1}_{\Pi^k_n}(x_n)$ is the indicator function of the set Π^k_n. By definition, the value of the dual objective function is equal to

$$W(p_0) = \sum_{\substack{n=1\ldots N \\ k=0\ldots K}} \int_{D^k_n} p^k_n(x_n) f(x_n)dx_n \tag{2.6.14}$$

or, if I take the summation with respect to k inside the integral, we can rewrite it as

$$W(p_0) = \sum_{n=1\ldots N} \int_{X_n} \left(\sum_{k=0\ldots K} p^k_n(x_n) \mathbf{1}_{D^k_n}(x_n) \right) f_n(x_n)dx_n \tag{2.6.15}$$

where $\mathbf{1}_{D^k_n}(x_n)$ is the indicator function of the set D^k_n. By definition of the demand sets, $p^k_n(x_n) \geq p^l_n(x_n)$ for any $x_n \in D^k_n$, $l = 0, \ldots K$. Therefore, $\sum_{k=0\ldots K} p^k_n(x_n)\mathbf{1}_{D^k_n}(x_n) \geq \sum_{k=0\ldots K} p^k_n(x_n)\mathbf{1}_{\Pi^k_n}(x_n)$. This proves the first part of the theorem.

(ii) Suppose that $\widehat{p_0}$ is the solution of 2.3.21. If the conditions of the theorem hold then the function $W(p_0)$ is differentiable at point $\widehat{p_0}$ and the explicit expression for the derivative has been derived in section 2.3.4. It has been shown that

$$\text{for each } k = 1, \ldots, K \quad \mu_n(D_n^k) = \mu_{\tilde{n}}(D_{\tilde{n}}^k) \quad \text{for any } n, \tilde{n} \tag{2.6.16}$$

where D_n^k are the demand sets that correspond to the optimal transfers $\widehat{p_0}$. Therefore, the demand sets generate a partition $\widehat{\mathcal{P}}$, $\widehat{\Pi}_n^k = D_n^k$, of the sets X_n that satisfies the constraints of the problem 2.3.18. From 2.6.13, 2.6.15, and the fact that $\widehat{\Pi}_n^k = D_n^k$ I obtain that $W(\widehat{p_0}) = V(\widehat{\mathcal{P}})$. ¿From part (i) of the lemma it follows immediately that $\widehat{\mathcal{P}}$ is the solution of problem 2.3.18.

Finally, if $\widehat{p_0}$ is a solution to 2.3.21 then it must satisfy the constraints of 2.3.21, therefore, the surplus balance condition holds, and the first order conditions 2.6.16 must hold. The equations 2.6.16 are the mass balance conditions and, therefore, the transfers $\widehat{p_0}$ and the corresponding demand sets D_n^k describe the equilibrium of the model.

(iii) Suppose that the equilibrium of the model exists. Let $\widehat{p_0}$ be the equilibrium transfers and D_n^k be the corresponding demand sets. From the surplus balance condition it follows that the transfers $\widehat{p_0}$ satisfy the constraints of the problem 2.3.21. From the mass balance condition it follows that the partition $\widehat{\mathcal{P}}$ that corresponds to the demand sets D_n^k satisfies the constraints of problem 2.3.18. From 2.6.13 and 2.6.15 it follows that $W(\widehat{p_0}) = V(\widehat{\mathcal{P}})$.

■

Theorem 2.3.2 Equilibrium of $\mathcal{M}_N^{\mathcal{K}}$ always exists.

Proof. In Proposition 2.3.2 I have shown existence of the equilibrium in the case that Assumption 2.3.1 holds. Now I show that the equilibrium of $\mathcal{M}_N^{\mathcal{K}}$ exists for an arbitrary array a_n^k. Suppose that a_n^k is some arbitrary array of parameters of the model and $\{a_n^{k,(j)}\}_{j=0,1,2,\dots}$ is a sequence of arrays of parameters of the model such that for each j Assumption 2.3.1 holds for the array $a_n^{k,(j)}$ and $a_n^{k,(j)} \to a_n^k$ for any n and k as $j \to \infty$. By Proposition 2.3.2, there exists an equilibrium, described by an array of prices $\widehat{p}_n^{k,(j)}$ and corresponding demand sets $D_n^{k,(j)}$, in the model with parameters $a_n^{k,(j)}$. Without loss of generality, I assume that for each n and k the sequences $\widehat{p}_n^{k,(j)}$ and $\mu_n(D_n^{k,(j)})$ converge[37]. Let $\widehat{p}_n^k$ and μ_n^k denote the limits of the sequences.

The transfers $\widehat{p}_n^k$ generate the upper demand sets $\overline{D}_n^k$ as described in 2.3.8. Since the distribution μ_n has a continuous density function then for any pair of decisions (k,l) either $\overline{D}_n^k = \overline{D}_n^l$ or $\mu_n(\overline{D}_n^k \cap \overline{D}_n^l) = 0$. The case $\overline{D}_n^k = \overline{D}_n^l$ occurs whenever $a_n^k = a_n^l$ and $\widehat{p}_n^k = \widehat{p}_n^l$. Let $\mathcal{K}(k)$ denote the set of decisions $\{l : \overline{D}_n^k = \overline{D}_n^l\}$.

Let $D_n^{\mathcal{K}(k),(j)} = \bigcup_{l\in\mathcal{K}(k)} D_n^{l,(j)}$ and suppose that x_n belongs to the interior of the set $\overline{D}_n^k$. Then, by definition of the upper demand set, $a_n^k x_n + \widehat{p}_n^k > a_n^l x_n + \widehat{p}_n^l$ for any $l \notin \mathcal{K}(k)$. Therefore, if j is large enough, then $a_n^{k,(j)} x_n + \widehat{p}_n^{k,(j)} > a_n^{l,(j)} x_n + \widehat{p}_n^{l,(j)}$ for any $l \notin \mathcal{K}(k)$. The last inequality implies that x_n belongs to the interior of $D_n^{\mathcal{K}(k),(j)}$ for j large enough. Analogously, if x_n belongs to the interior of the complement of $\overline{D}_n^k$ then x_n belongs to the interior of the complement of the set $D_n^{\mathcal{K}(k),(j)}$. Since measure μ_n is continuous the measure of the boundary of the set $\overline{D}_n^k$ is zero. Therefore,

$$\mu_n(\overline{D}_n^k) = \sum_{l\in\mathcal{K}(k)} \mu_n^l \tag{2.6.17}$$

[37] If the sequence does not converge for some n and k I can always choose a converging subsequence.

The demand sets D_n^l are constructed now as follows. For each subset of indices $\mathcal{K}(k)$ of the set of indices $\{0, 1, \ldots, K\}$ I construct an arbitrary partition of the set $\overline{D}_n^k$ into measurable subsets D_n^l, $l \in \mathcal{K}(k)$ such that $\mu_n(D_n^l) = \mu_n^l$. It is straightforward to show now that the array of prices $\widehat{p}_n^k$ and demand sets D_n^l describe the equilibrium of the model with parameters a_n^k.

■

2.6.3 Eigenvectors of $\Omega_n(p_{n,0})$ when Eigenvalue is Zero

Lemma 2.3.2 Vectors $\upsilon_{n,i}$, $i = 1 \ldots \eta$ are eigenvectors that correspond to the eigenvalue of zero of Ω_n.

Proof. I need to show that for any row k the following equality holds: $\sum_{l=1}^{K} \omega_n^{kl} \upsilon_{n,i}^l = 0$. It follows from the definition of vector $\upsilon_{n,i}^l$ that

$$\sum_{l=1}^{K} \omega_n^{kl} \upsilon_{n,i}^l = \sum_{l \in \mathcal{I}_{n,i}} \omega_n^{kl}$$

So it is sufficient to show that $\sum_{l \in \mathcal{I}_{n,i}} \omega_n^{kl}$. Consider, first, $k \notin I_{n,i}$. For any row k any column l for which $\omega_n^{kl} \neq 0$ must be adjacent to k and, therefore, k and l must belong to a common set $I_{n,j}$ for some j. Consider $I_{n,i}$ in 2.3.27.

If $k \notin I_{n,i}$ then k is not adjacent to any $l \in I_{n,i}$. Therefore, $\omega_n^{kl} = 0$ for any $k \notin I_{n,i}$, $l \in I_{n,i}$ and $\sum_{l \in \mathcal{I}_{n,i}} \omega_n^{kl} = 0$ for any $k \notin I_{n,i}$.

If $k \in I_{n,i}$ then k is not adjacent to any $l \notin I_{n,i}$. Therefore, $\omega_n^{kl} = 0$ for any $k \in I_{n,i}$, $l \notin I_{n,i}$ and

$$\sum_{l \in \mathcal{I}_{n,i}} \omega_n^{kl} = \sum_{l=1}^{K} \omega_n^{kl}$$

for any $k \in I_{n,i}$. By 2.3.26 I obtain $\sum_{l=1}^{K} \omega_n^{kl} = 0$. ■

Lemma 2.3.3 Let matrix Ω_n be given by formula 2.3.26. Then the vectors, defined by 2.3.27, form a basis of the space of eigenvectors that correspond to the eigenvalue of zero.

Proof. Let υ be such that $\Omega_n \upsilon = 0$. To show that υ can be represented as a linear combination of $\upsilon_{n,i}$, $i = 1 \ldots \eta$ vectors it is sufficient to show that $\upsilon^k = \upsilon^l$ for any $k, l \in I_{n,i}$. Let $\upsilon^{k_1} = \max_{j \in \mathcal{I}_{n,i}} \upsilon^j$.

¿From the formula of the second derivative 2.3.26 I obtain $\omega_n^{k_1 k_1} = -\sum_{k:k \neq k_1} \omega_n^{k_1 k}$. For any $k \notin I_{n,i}$ column k is not connected to k_1 and, therefore, $\omega_n^{k_1 k} = 0$. Thus the equality can be rewritten as $\omega_n^{k_1 k_1} = -\sum_{\substack{k \in \mathcal{I}_{n,i} \\ k \neq k_1}} \omega_n^{k_1 k}$. Since $-\omega_n^{k_1 k} \overset{2.3.26}{=} \tau_n^{k_1 k} \geq 0$ and $\upsilon^{k_1} \geq \upsilon^k$ for any $k \in I_{n,i}$ and $k \neq k_1$ I obtain

$$\omega_n^{k_1 k_1} \upsilon^{k_1} \geq - \sum_{\substack{k \in \mathcal{I}_{n,i} \\ k \neq k_1}} \omega_n^{k_1 k} \upsilon^k \tag{2.6.18}$$

and the inequality is strict if $\upsilon^{k_1} > \upsilon^k$ for some $k \neq k_1$ such that $-\omega_n^{k_1 k} > 0$. On the other hand, since $\Omega_n \upsilon = 0$ inequality 2.6.18 must be an equality. Therefore $\upsilon^k = \upsilon^{k_1}$ for any k such that $\omega_n^{k_1 k} > 0$. This proves that $\upsilon^k = \upsilon^{k_1}$ for any column k adjacent to k_1. Applying the same argument to any column k adjacent to k_1 I can show that $\upsilon^l = \upsilon^k$ for any column l adjacent to k since $\upsilon^k = \max_{j \in \mathcal{I}_{n,i}} \upsilon^j$. But then it must be true that $\upsilon^l = \upsilon^{k_1}$ for any column $l \in I_{n,i}$ connected to k_1. ■

2.6.4 Uniqueness of the Equilibrium

Lemma 2.3.4 Suppose that $\widetilde{p}_0$ and $\widehat{p}_0$ are two equilibrium transfer matrices and k and l are two decisions that are connected in $\Omega_n(\widetilde{p}_{n,0})$ at transfers $\widetilde{p}_{n,0}$. Then $\widehat{p}_{n,0}^k - \widetilde{p}_{n,0}^k = \widehat{p}_{n,0}^l - \widetilde{p}_{n,0}^l$.

Proof. If $\widetilde{p}_0$ and $\widehat{p}_0$ are two equilibrium transfer matrices then they are solutions to the minimization problem 2.3.21 and $W(\widetilde{p}_0) = W(\widehat{p}_0)$. Moreover, by convexity of $W(p_0)$, any transfer matrix $p_0^t = \widetilde{p}_0 + t(\widehat{p}_0 - \widetilde{p}_0)$, $t \in [0,1]$ is a solution to 2.3.21 and the function $s(t) = W(p_0^t)$ is constant for $t \in [0,1]$. Therefore the second derivative of $s(t)$ is zero.

I now find the second derivative of $s(t)$. By 2.3.19 $W(p_0^t) = \sum_m W_m(p_{m,0}^t)$. By 2.3.26 the second derivative of $W_m(p_{m,0}^t)$ is $\Omega_m(p_{n,0}^t)$. Therefore, the second derivative of $s(t)$ is

$$\frac{d^2 s(t)}{dt^2} = \sum_{m=1}^{N} (\widehat{p}_{m,0} - \widetilde{p}_{m,0})^T \Omega_m(p_{m,0}^t)(\widehat{p}_{m,0} - \widetilde{p}_{m,0}) \tag{2.6.19}$$

Therefore, since each function $W_m(p_{m,0})$ is convex (Proposition 2.3.3) the second derivative $\frac{d^2 s(t)}{dt^2} = 0$ only if $(\widehat{p}_{m,0} - \widetilde{p}_{m,0})^T \Omega_m(p_{m,0}^t)(\widehat{p}_{m,0} - \widetilde{p}_{m,0}) \equiv 0$ for each m and $t \in [0,1]$. In particular, if I take $m = n$

$$(\widehat{p}_{n,0} - \widetilde{p}_{n,0})^T \Omega_n(p_{n,0}^t)(\widehat{p}_{n,0} - \widetilde{p}_{n,0}) \equiv 0$$

where superscript T denotes transposition. Therefore, either $\widehat{p}_{n,0} - \widetilde{p}_{n,0} = 0$ or $\widehat{p}_{n,0} - \widetilde{p}_{n,0}$ is an eigenvector of $\Omega_n(p_{n,0}^t)$ that corresponds to the eigenvalue of zero for any $t \in [0,1]$.

In the first case we are done. In the second case $\widehat{p}_{n,0} - \widetilde{p}_{n,0}$ is eigenvector of $\Omega_n(p_{n,0}^t)|_{t=0} = \Omega_n(\widetilde{p}_{n,0})$ that corresponds to the eigenvalue of zero. By corollary 2.3.1, I obtain $\widehat{p}_{n,0}^k - \widetilde{p}_{n,0}^k = \widehat{p}_{n,0}^l - \widetilde{p}_{n,0}^l = 0$ for any connected decisions $k, l \in \mathcal{K}^+(\widetilde{p}_0)$. This proves the lemma. ■

Lemma 2.3.5 Suppose that measure μ_n has a continuous density function $f_n(x)$. If a set of agents types $Y \in X_n$ is connected[38] with respect to μ_n and decisions k and l are such that $\mu_n(Y \cap D_n^k) > 0$ and $\mu_n(Y \cap D_n^l) > 0$,

[38]The definition of a set connected with respect to μ_n is given on page 31

where demand sets D_n^k and D_n^l are constructed for some transfer vector $p_{n,0}$, then decisions k and l are connected[39] in $\Omega_n(p_{n,0})$ at transfers $p_{n,0}$.

Proof. Suppose that decisions k and l are not connected. By definition, $\tau_n^{\widetilde{k}\widetilde{l}} \overset{2.3.26}{=} -\omega_n^{\widetilde{k}\widetilde{l}} = 0$ for any decision $\widetilde{k}$ that is connected to k and any decision $\widetilde{l}$ that is not connected to k. From Definition 2.3.25 of $\tau_n^{\widetilde{k}\widetilde{l}}$ I obtain $f_n(x_n) = 0$ for any $x_n \in \Gamma_n^{\widetilde{k}\widetilde{l}}$ since $f_n(x_n)$ is continuous.

Define the set $Y^k \in Y$ as the set of agents types in the set Y that make decision k or any decision that is connected to k in $\Omega_n(p_{n,0})$. Let $\mathcal{K}^k$ denote the set of such decisions. Define the set $Y^l \in Y$ as the set of agents types in the set Y that make decision l or any decision that is not connected to decision k in $\Omega_n(p_{n,0})$. Let $\mathcal{K}^l$ denote the set of such decisions.

By construction, $Y = Y^k \cup Y^l$ and the sets $A = Y^k \backslash \partial Y^k$ and $B = Y^l \backslash \partial Y^l$ are disjoint (where symbol ∂ denotes the boundary of a set). By assumption, $\mu_n(A) > 0$ and $\mu_n(B) > 0$. On the other hand,

$$Y \backslash (A \cup B) \subset \cup_{\substack{\widetilde{k} \in \mathcal{K}^k \\ \widetilde{l} \in \mathcal{K}^l}} \Gamma_n^{\widetilde{k}\widetilde{l}}$$

and, therefore, $f_n(x_n) = 0$ for any $x_n \in Y \backslash (A \cup B)$. This contradicts the assumption that set the Y is connected with respect to μ_n. ■

Theorem 2.3.3 Suppose that Assumption 2.3.1 holds and in each class n there is a continuum of agents endowed with multi-dimensional types and the distribution of types μ_n has a continuous density functions $f_n(x)$. Then the following statements are true.

(i) The set of partitions, generated by the set of solutions of the dual minimization problem 2.3.21, is a singleton.

[39] See page 29 for the definition of decisions connected in $\Omega_n(p_{n,0})$.

(ii) Suppose that for some n, the set of types X_n is connected with respect to μ_n. Also suppose that the transfers in the case of decision k_0 satisfy $\widetilde{p}^{k_0}_{n,0} = \widehat{p}^{k_0}_{n,0}$ for some decision $k_0 \in \mathcal{K}^+(\widehat{p}_0)$. Then $\widetilde{p}^{k}_{n,0} = \widehat{p}^{k}_{n,0}$ for any $k \in \mathcal{K}^+(\widehat{p}_0)$.

(iii) For any class n the vector of optimal transfers $\widehat{p}_{n,0}$ is determined uniquely if the set of types X_n is connected with respect to μ_n and a positive measure of agents are unmatched in class n.

Proof.

(i) Suppose that $\widehat{p}_0$ and $\widetilde{p}_0$ are two solutions of 2.3.21 that generate set partition vectors $\widehat{\mathcal{P}} = \left(\widehat{\mathcal{P}}_1, \ldots, \widehat{\mathcal{P}}_N\right)$ and $\widetilde{\mathcal{P}} = \left(\widetilde{\mathcal{P}}_1, \ldots, \widetilde{\mathcal{P}}_N\right)$. Suppose that $\widehat{\mathcal{P}} \neq \widetilde{\mathcal{P}}$ so that for some n, there exists a ball B^{ϵ}_n in set X_n that has a positive measure and is connected with respect to μ_n, and such that $B^{\epsilon}_n \in D^{\widehat{k}}_n$ for some decision $\widehat{k}$, where $D^{\widehat{k}}_n$ is a subset of a partition $\widehat{\mathcal{P}}_n$, and $B^{\epsilon}_n \in D^{\widetilde{k}}_n$ for some decision $\widetilde{k} \neq \widehat{k}$, where $D^{\widetilde{k}}_n$ is a subset of a partition $\widetilde{\mathcal{P}}_n$.

By convexity of $W(p_0)$, any transfer matrix $p^t_0 = t\widehat{p}_0 + (1-t)\widetilde{p}_0, \quad t \in [0,1]$ is a solution to problem 2.3.21 and, therefore, the transfers p^t_0 and the corresponding demand sets $D^{t,k}_n$ are an equilibrium of the model. As t goes from zero to one the demand set $D^{t,\widehat{k}}_n$ changes continuously. At $t = 0$ $B^{\epsilon}_n \in D^{t=0,\widehat{k}}_n$ and at $t = 1$ $B^{\epsilon}_n \in D^{t=1,\widehat{k}}_n$. Therefore, there exists a value of $t = t_0$ and a decision $k \neq \widehat{k}$ such that a positive measure of agents' types that belong to B^{ϵ}_n choose decision $\widehat{k}$ and a positive measure of agents' types that belong to B^{ϵ}_n choose decision k. By Lemma 2.3.5 the decisions $\widehat{k}$ and m are connected at transfers $p^{t_0}_0$. This can be illustrated by the following figure.

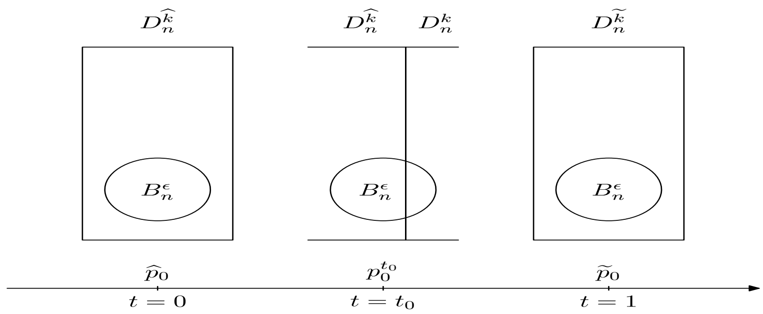


Figure 2.6.1 In the figure any agent in class n of type $x_n \in B_n^\epsilon$ makes decision $\widehat{k}$ at transfers $\widehat{p}_0$. If the transfers are $p_0^{t_0}$ then a positive measure of agents in class n of types $x_n \in B_n^\epsilon$ make decision $\widehat{k}$ and a positive measure of agents in class n of types $x_n \in B_n^\epsilon$ make decision k. If the transfers are $\widetilde{p}_0$ then agents in class n of types $x_n \in B_n^\epsilon$ make decision $\widetilde{k}$.

By Lemma 2.3.4, I obtain $\widehat{p}_{n,0}^{\widehat{k}} - p_{n,0}^{t_0,\widehat{k}} = \widehat{p}_{n,0}^{k} - p_{n,0}^{t_0,k}$. But this implies that any agent in class n who prefers decision k to $\widehat{k}$ at transfers $p_{n,0}^{t_0}$ will also prefer decision k to $\widehat{k}$ at transfers $p_{n,0}^{\widehat{k}}$. This contradicts the fact that any agent in class n of type $x_n \in B_n^\epsilon$ most prefers decision $\widehat{k}$ at transfers $\widehat{p}_{n,0}$.

(ii) By Lemma 2.3.5, if the set X_n is connected then any two decisions that are taken by a positive measure of agents in class n and are connected. In particular, any such decision is connected to decision k_0. By lemma 2.3.4, $\widehat{p}_{n,0}^{k} - \widehat{p}_{n,0}^{k_0} = \widetilde{p}_{n,0}^{k} - \widetilde{p}_{n,0}^{k_0}$ for any two equilibrium transfers $\widehat{p}_0$ and $\widetilde{p}_0$. In particular, if $\widehat{p}_{n,0}^{k_0} = \widetilde{p}_{n,0}^{k_0}$ then $\widehat{p}_{n,0}^{k} = \widetilde{p}_{n,0}^{k}$.

(iii) The transfer for the decision to remain unmatched is always constrained to be r_n^0, $\widehat{p}_{n,0}^{0} = \widetilde{p}_{n,0}^{0} = r_n^0$. As above, lemmas 2.3.5 and 2.3.4 imply that $\widehat{p}_{n,0}^{k} = \widetilde{p}_{n,0}^{k}$ for any decision $k \in \mathcal{K}^+(p_0)$.

■

2.6.5 Alternative Definition of the Equilibrium

In this section, I consider a matching model which is identical to the model of section 2.3.1 but in which the definition of equilibrium does not depend on Assumption 2.3.1. The definition is a natural extension of Definition 2.3.1.

Let's consider again the matching model of section 2.3.1. Suppose that the surplus generated in a match is allocated among the matched agents and the portion of the surplus received by an agent of type x_n is of the form given in 2.3.6 and 2.3.7. Now, in contrast with the equilibrium definition of section 2.3.2, I associate a demand set with each subset of the set of decisions $\mathcal{K}$. Let $2^{\mathcal{K}}$ denote the set of all subsets of $\mathcal{K}$ and let S denote an elements of $2^{\mathcal{K}}$. Then the demand set associated with a subset S of the set $\mathcal{K}$ is defined as follows.

$$D_n^S = \Big\{ x_n \in X_n : p_n^k(x_n) \geq p_n^l(x_n) \quad \text{for any } k \in S \text{ and } l = 0, \ldots, K \quad \text{and} \\ p_n^k(x_n) = p_n^{\widetilde{k}}(x_n) \quad \text{for any } k, \widetilde{k} \in S \Big\} \quad (2.6.20)$$

That is, the set D_n^S is the set of types of agents in class n who strictly prefer a decision $k \in S$ to any decision that is not in S and who are indifferent among the decisions in the set S. Let ν_{nk}^S denote the mass of agents in class n of type $x_n \in D_n^S$ that choose decision $k \in S$. By definition of ν_{nk}^S, the collection ν_{nk}^S must satisfy

$$\sum_{k \in S} \nu_{nk}^S = 1 \quad \text{for any } S \in 2^{\mathcal{K}} \quad (2.6.21)$$

I define the equilibrium of the model as follows.

Definition 2.6.1 *A matrix of transfers* $p_{n,0}^k$*, corresponding demand sets* $D_n^S\left(p_{n,0}\right)$*, and a collection* ν_{nk}^S *is an equilibrium if:*

1. *The measure of the agents that choose decision k is the same in each set X_n. That is, for each $k = 1, \dots, K$*

$$\sum_{S \in 2^{\mathcal{K}}} \sum_{k \in S} \mu_n \left(D_n^S\right) \nu_{nk}^S = \sum_{S \in 2^{\mathcal{K}}} \sum_{k \in S} \mu_{\widetilde{n}} \left(D_{\widetilde{n}}^S\right) \nu_{\widetilde{n}k}^S \qquad \text{for any } n \text{ and } \widetilde{n}. \tag{2.6.22}$$

2. *The sum of transfers to the agents in a matched group equals the constant term in the surplus generated by the group.*

$$\text{for each } k = 1, \dots, K \quad \sum_n p_{n,0}^k = a_0^k \tag{2.6.23}$$

2.6.6 Equivalence of the Equilibrium Concepts

In this section, I show how the equilibrium of the model of section 2.4.2 can be naturally transformed into the equilibrium, introduced in Definition 2.6.1, of the model of section 2.3.1. (Formal proof of the equivalence of the two equilibrium concepts is beyond the scope of this work). I illustrate the equivalence of the two equilibrium concepts by considering the two equilibrium examples of section 2.4.2, the first example on page 47 (Figures 2.4.7 and 2.4.8) and the second example on page 48 (Figures 2.4.9 and 2.4.10).

To give an equivalent representation of the equilibrium in each of the two examples, I describe the corresponding transfers $p_n^{k_1,k_2,l_1,l_2}$, demand sets D_n^S, and parameters ν_{nk}^S. Let $\widehat{p}$ denote the equilibrium transfer in each of the two examples of section 2.4.2 and let $\mathbf{p} = (0, \widehat{p})$. The transfer $\widehat{p}$ is transformed into the following transfers $p_n^{k_1,k_2,l_1,l_2}$ that correspond to decisions (k_1, k_2, l_1, l_2).

$$\begin{cases} p_1^{k_1,k_2,l_1,l_2} &= b_1^{k_1,l_1} + b^{l_1,l_2} - \mathbf{p}^{l_2} \\ p_2^{k_1,k_2,l_1,l_2} &= b_2^{k_2,l_2} + \mathbf{p}^{l_2} \end{cases} \tag{2.6.24}$$

In both examples, there are only two non-empty demand sets D_n^S in each class $n = 1, 2$. Let's consider the first example (Figures 2.4.7 and 2.4.8). Let

D_n^a and D_n^b, $n = 1, 2$, denote the non-empty demand sets in the example. The four sets are

$$D_1^a = [0, \overline{x}_1] \qquad D_2^a = [0, \overline{x}_2]$$
$$D_1^b = [\overline{x}_1, \overline{\vartheta}_1] \qquad D_2^b = [\overline{x}_2, \overline{\vartheta}_2]$$

Let S_n^a and S_n^b, $n = 1, 2$, denote the sets of decisions that correspond to the demand sets in class n. The sets of decisions are

$$S_1^a = \{k : k = (k_1 = 0, k_2 - \text{any}, l_1 = 0, l_2 = 0)\}$$
$$S_1^b = \{k : k = (k_1 = 1, k_2 - \text{any}, l_1 = 1, l_2 = 1)\}$$

$$S_2^a = \{k : k = (k_1 - \text{any}, k_2 = 0, l_1 - \text{any}, l_2 = 0)\}$$
$$S_2^b = \{k : k = (k_1 - \text{any}, k_2 = 1, l_1 - \text{any}, l_2 = 1)\}$$

The corresponding non-zero[40] parameters ν_{nk}^S are

$$\nu_{1,(k_1=0,k_2=0,l_1=0,l_2=0)}^{S_1^a} = 1 \qquad \nu_{2,(k_1=0,k_2=0,l_1=0,l_2=0)}^{S_2^a} = 1$$
$$\nu_{1,(k_1=1,k_2=1,l_1=1,l_2=1)}^{S_1^b} = 1 \qquad \nu_{2,(k_1=1,k_2=1,l_1=1,l_2=1)}^{S_2^b} = 1$$

Let's consider now the second example (Figures 2.4.9 and 2.4.10). Let D_n^a and D_n^b, $n = 1, 2$, denote the non-empty demand sets in the example. The four sets are

$$D_1^a = [0, x_1^r] \qquad D_2^a = [0, \overline{x}_2]$$
$$D_1^b = [x_1^r, \overline{\vartheta}_1] \qquad D_2^b = [\overline{x}_2, \overline{\vartheta}_2]$$

Let S_n^a and S_n^b denote the sets of decisions that correspond to the demand sets in class n. The sets of decisions are

$$S_1^a = \{k : k = (k_1 = 0, k_2 - \text{any}, l_1 = 0, l_2 = 0)\}$$
$$S_1^b = \{k : k = (k_1 = 1, k_2 - \text{any}, l_1 = 1, l_2 - \text{any})\}$$

$$S_2^a = \{k : k = (k_1 - \text{any}, k_2 = 0, l_1 - \text{any}, l_2 = 0)\}$$
$$S_2^b = \{k : k = (k_1 - \text{any}, k_2 = 1, l_1 - \text{any}, l_2 = 1)\}$$

The corresponding non-zero parameters ν_{nk}^S are

$$\nu_{1,(k_1=0,k_2=0,l_1=0,l_2=0)}^{S_1^a} = 1 \qquad \nu_{2,(k_1=0,k_2=0,l_1=0,l_2=0)}^{S_2^a} = \frac{\mu_1([0,x_1^r])}{\mu_2([0,\overline{x}_2])}$$
$$\nu_{1,(k_1=1,k_2=0,l_1=1,l_2=0)}^{S_1^b} = \frac{\mu_2([0,\overline{x}_2])-\mu_1([0,x_1^r])}{\mu_1([x_1^r,\overline{\vartheta}_1])} \qquad \nu_{2,(k_1=1,k_2=0,l_1=1,l_2=0)}^{S_2^a} = \frac{\mu_2([0,\overline{x}_2])-\mu_1([0,x_1^r])}{\mu_2([0,\overline{x}_2])}$$
$$\nu_{1,(k_1=1,k_2=1,l_1=1,l_2=1)}^{S_1^b} = \frac{\mu_2([\overline{x}_2,\overline{\vartheta}_2])}{\mu_1([x_1^r,\overline{\vartheta}_1])} \qquad \nu_{2,(k_1=1,k_2=1,l_1=1,l_2=1)}^{S_2^b} = 1$$

[40]For any other decisions in the sets the parameters are zero.

In both cases it can be directly verified that the demand sets correspond to the transfers given in 2.6.24 and that the mass balance 2.6.22 condition holds.

Chapter 3

Extensions of $\mathcal{M}_{N=2}^{\mathcal{K}}$

3.1 Introduction

In this Chapter, I extend the N-lateral K-decision matching model introduced in Chapter 2 in two directions by relaxing some of the restrictions of the model when $N = 2$. In each extension of the bilateral model, I consider a bilateral matching model with K decisions. In the first extension of the bilateral model, the surplus generated in a match has a more general form than that considered in the first essay. I derive some sufficient conditions and some necessary conditions under which there exists an equilibrium matching which is positive assortative. Specifically, I consider a matching model in which there are two classes of agents, the types of the agents are one-dimensional, and the surplus generated in a match has a general form as a function of the types of the agents and the decision made in the match (in particular, it may be non-separable in the types of the agents). I show that if (1) for any decision the types of the agents are complementary to each other and (2) the type of an agent is complementary to the decisions made in the match, then there exists an equilibrium matching that is positive assortative. In the second extension of the bilateral model, the utility of an agent in a match has a more general form than that considered in the first essay. Specifically, the utility of an agent in a match depends on his type, the type of his partner, decision made in the match, and some random component. I show that, under certain conditions, the equilibrium exists and is unique and I discuss how the equilibrium can be constructed numerically.

3.2 Assortative Matching

The previous bilateral matching literature offers some sufficient conditions that guarantee positive assortative matching in equilibrium. In this section, I provide some necessary conditions that are implied by positive assortative matching as well as some sufficient conditions that guarantee positive assortative matching in a bilateral matching model that extends the bilateral model $\mathcal{M}^{\mathcal{K}}_{N=2}$ introduced in chapter 2. Thus, I derive the analogue of results from the previous bilateral matching literature in the case that the surplus function depends on the matched types as well as the decision made in the match.

In contrast with chapter 2, I assume in this section that there are only $N = 2$ classes of agents and that the types of the individuals are one-dimensional. In common with chapter 2, there is a continuum of types of agents in each class and the agent's choice of a partner depends on the decision made in the match. I extend the bilateral model by allowing that the choice of a partner depends also on the types of the matched individuals.

I find that the following conditions are sufficient to imply that positive assortative matching maximizes the aggregate surplus.

- For each decision the surplus function is supermodular.
- The extra surplus generated by increasing an agent's type increases in decisions.

The first condition can be interpreted as complementarity between the types of the agents conditional on a given decision. The second condition can be interpreted as complementarity between the type of an agent and the decision made in the match. I also provide necessary conditions that are implied when positive assortative matching maximizes the aggregate surplus.

3.2.1 Model Set-Up

In this section, I describe a bilateral one-to-one matching problem with transferrable utility and one-dimensional types of agents. Let $x \in \mathbb{R}$ denote the type of a male and $y \in \mathbb{R}$ denote the type of a female. The distribution of male types is denoted by μ_X and that of female types by μ_Y. A matching between male and female types is a function $m : \mathbf{R} \rightarrow \mathbf{R}$ that satisfies the following mass balance condition

$$\text{for any } E \subset \mathbf{R} : \quad \mu_1(m^{-1}(E)) = \mu_2(E) \tag{3.2.1}$$

Whenever $y = m(x)$ I interpret it as a match between a type x male and a type y female. If a male of type x is matched to a female of type y, the pair generates a surplus $u(x, y)$. Results from the standard bilateral matching literature (for example, [2]) guarantee that equilibrium matching $m(\cdot)$ can be constructed as a solution of the following planner's constrained maximization problem

$$W = \max_{m(\cdot)} \int_{\mathbf{R}} u(x, m(x)) d\mu_1(x) \tag{3.2.2}$$

subject to the constraint 3.2.1 on the matching function. Objective function W is the aggregate surplus generated in all matches.

I consider a surplus function of the form

$$u(x, y) = \max_{k=1}^{K} [u^k(x, y)] \tag{3.2.3}$$

That is, the surplus depends in a general way on both the types of the agents and the decision made in the match.

A matching $m(\cdot)$ is positive assortative if $m(\cdot)$ is a nondecreasing function of x. In the next two sections I derive sufficient conditions and necessary conditions for positive assortative matching to maximize the aggregate surplus function.

3.2.2 Sufficient Conditions for Positive Assortative Matching

In this section, I derive sufficient conditions that guarantee that positive assortative matching maximizes the aggregate surplus. In [2] it is shown that one such sufficient condition is supermodularity of the surplus function. A surplus function is supermodular if for any $x'' > x'$ and $y'' > y'$

$$u(x'', y'') + u(x', y') \geq u(x'', y') + u(x', y'')$$

The supermodularity of the functions $u^k(x, y)$ does not guarantee supermodularity of the function $u(x, y)$. To describe sufficient conditions for the supermodularity of $u(x, y)$ I introduce first the following definitions. Let A^{kl} denote the set of pairs (x, y) such that

$$A^{kl} = \left\{(x, y) \in \mathbb{R}^2 : u^k(x, y) \geq u^l(x, y)\right\} \tag{3.2.4}$$

That is, A^{kl} is the set of pairs (x, y) of agent-types that prefer decision k over decision l if matched. The following assumption imposes some regularity conditions on the sets A^{kl}.

Assumption 3.2.1 *For any x and any pair (k, l) there exists a unique solution $y = y_{kl}(x)$ of the equation*

$$u^k(x, y) = u^l(x, y) \tag{3.2.5}$$

For each k and l, the function $y_{kl}(x)$ is a continuous function of x.

The pairs $(x, y_{kl}(x))$ describes the boundary between the sets A^{kl} and A^{lk}. Lemma provides sufficient conditions for supermodularity of the function $u(x, y)$.

Lemma 3.2.1 *Suppose that Assumption 3.2.1 holds. If for any $k = 1 \dots K$, the function $u^k(x, y)$ is supermodular and the functions $y_{kl}(x)$ are nonincreasing in x for all k and l, then the function $u(x, y) = \max_{k=1}^{K} u^k(x, y)$ is supermodular and positive assortative matching maximizes the aggregate surplus.*

The proof is in the Appendix[1].

In the case that the components of the surplus function satisfy

$$u^k(x, y) = a_1^k x + a_2^k y + a_0^k \tag{3.2.6}$$

the conditions of Lemma 3.2.1 can be simplified. Each function $u^k(x, y)$ is supermodular since the second partial cross derivative is zero[2]. Therefore, the surplus function is supermodular when each function $y = y_{kl}(x)$, defined in 3.2.5, is nonincreasing with respect to x. When the surplus functions satisfy 3.2.6 the equation 3.2.5 is satisfied when

$$a_1^k x + a_2^k y + a_0^k = a_1^l x + a_2^l y + a_0^l \tag{3.2.8}$$

[1]The result is proved only for a finite number of decisions. However, it can be easily extended for the case when $k \in [\underline{k}, \overline{k}]$. Suppose that the surplus function is $u(x_1, x_2) = \max_{k \in [\underline{k}, \overline{k}]} u(x_1, x_2, k)$, where $u(x_1, x_2, k)$ is continuous in its arguments. The function can be approximated arbitrary close by a function $u^n(x_1, x_2) = \max_{k \in \mathcal{K}_n} u(x_1, x_2, k)$, where $\mathcal{K}_n \in [\underline{k}, \overline{k}]$ is some finite set of elements. Applying Lemma 3.2.1 to the functions $u^n(x_1, x_2)$ and taking the limit I can prove the result for the function $u(x_1, x_2)$.

[2]A standard sufficient condition for the supermodularity of a surplus function $u(x, y)$ is

$$\frac{d^2 u(x, y)}{dx dy} \geq 0 \tag{3.2.7}$$

From equation 3.2.8 I find

$$y_{kl}(x) = \frac{a_1^l - a_1^k}{a_2^k - a_2^l}x + \frac{a_0^l - a_0^k}{a_2^k - a_2^l} \tag{3.2.9}$$

From 3.2.9 it follows that for any k and l the functions $y_{kl}(x)$ are non-increasing if and only if the following condition is satisfied. For any indices $k > l$

$$a_1^k \geq a_1^l \quad \text{and} \quad a_2^k \geq a_2^l \tag{3.2.10}$$

Thus, in the case that 3.2.6 holds, positive assortative matching follows from 3.2.10. When 3.2.6 fails to hold, an analogous result hold if 3.2.10 is replaced with inequalities of the partial derivatives of u^k.

Proposition 3.2.1 *Suppose that Assumption 3.2.1 holds. Suppose also that for each k the function $u^k(x, y)$ is supermodular and that for any indices $k > l$*

$$\frac{\partial u^k}{\partial x}(x, y) \geq \frac{\partial u^l}{\partial x}(x, y) \quad \textit{and} \quad \frac{\partial u^k}{\partial y}(x, y) \geq \frac{\partial u^l}{\partial y}(x, y) \tag{3.2.11}$$

for all x and y. Then the surplus function $u(x, y)$ is supermodular and the positive assortative matching maximizes the aggregate surplus.

The proof is in the Appendix. Proposition 3.2.1 also follows from the results in Topkins, D., 1998 ([36], chapter 2, Theorem 2.7.6). I interpret condition 3.2.11 in Proposition 3.2.1 as complementarity between each type and decisions. More precisely, suppose that an increase in index $k = 1, \ldots, K$ corresponds to some ordering of the set of decisions from low level to high level decisions. Then condition 3.2.11 means that as the level of decision increases, the marginal change in surplus that corresponds to a marginal change in type also increases.

3.2.3 Necessary Conditions for Positive Assortative Matching

In this section I provide necessary conditions for positive assortative matching to maximize the aggregate surplus. The conditions demonstrate the idea that both complementary between types and complementary between each type and decisions are important in Proposition 3.2.1. I note first that positive assortative matching may maximize the aggregate surplus even when the surplus function $u(x, y)$ is submodular at some points (x, y). However, I show in Lemma 3.2.2 that the surplus maximizing matching is positive assortative then the surplus function must be supermodular at the points (x, y) such that $y = m(x)$.

Lemma 3.2.2 *Suppose that the measures* μ_X *and* μ_Y *are continuous and that the matching* $y = m(x)$ *maximizes the aggregate surplus. Suppose further that there exists a matched pair* $\tilde{o} = (\tilde{x}, \tilde{y})$ *such that* $\tilde{y} = m(\tilde{x}$ *and values* $x^a < \tilde{x}$ *and* $x^b > \tilde{x}$ *such that for any* $x^- \in [x^a, \tilde{x})$ *, and* $x^+ \in \left(\tilde{x}, x^b\right]$ *the following inequality holds:*

$$u\left(x^-, y^-\right) + u\left(x^+, y^+\right) < u\left(x^-, y^+\right) + u\left(x^+, y^-\right) \tag{3.2.12}$$

where $y^- = m(x^-)$ *and* $y^+ = m(x^+)$. *Then* $y = m(x)$ *is not positive assortative.*

The proof is in the Appendix. Next I show that, under some regularity conditions, if the types are complementary conditional on decisions but the functions y_{kl} are increasing for all k, l (so that complementarity between each type and decisions fails), then positive assortative matching does not maximize the aggregate surplus. First, I impose some regularity conditions on the surplus functions.

Assumption 3.2.2 *Suppose that the functions $u^k(x,y)$ are continuously differentiable for each k. Suppose also that for any k and l neither of the following systems has a solution.*

$$\begin{cases} u^k(x,y) = u^l(x,y) \\ \frac{\partial u^k(x,y)}{\partial x} = \frac{\partial u^l(x,y)}{\partial x} \end{cases} \quad or \quad \begin{cases} u^k(x,y) = u^l(x,y) \\ \frac{\partial u^k(x,y)}{\partial y} = \frac{\partial u^l(x,y)}{\partial y} \end{cases} \tag{3.2.13}$$

Assumption 3.2.2 rules out the curves y_{kl} that have slope either zero or infinity at some points so that functions y_{kl} similar to those illustrated in Figure 3.2.1 are not possible.

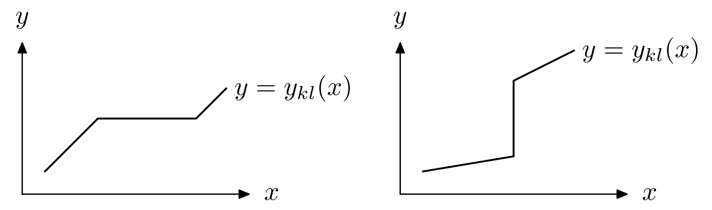


Figure 3.2.1 *Examples of $y_{kl}(x)$ functions that are ruled out by Assumption 3.2.2.*

Now I am ready to formulate the result that gives sufficient conditions when positive assortative matching does not maximize the aggregate surplus.

Proposition 3.2.2 *Suppose that Assumption 3.2.2 holds and that the functions $y_{kl}(x)$ are increasing functions of x. Suppose also that the measures μ_X and μ_Y are continuous. Then positive assortative matching $y = m(x)$ maximizes the aggregate surplus only if a common decision is made by all the matched pairs.*

The proof is in the Appendix.

3.3 Appendix

3.3.1 Positive Assortative Matching: Sufficient Conditions

Lemma 3.2.1 *Suppose that Assumption 3.2.1 holds. If for any* $k = 1 \ldots K$*, the function* $u^k(x, y)$ *is supermodular and the functions* $y_{kl}(x)$ *are nonincreasing in* x *for all* k *and* l*, then the function* $u(x, y) = \max_{k=1}^{K} u^k(x, y)$ *is supermodular and positive assortative matching maximizes the aggregate surplus.*

Proof. I prove Lemma 3.2.1 by induction. If $K = 1$ then the result is obviously true. Suppose that the result is shown for the utility function $u(x, y) = \max_{k=1}^{\tilde{K}} u^k(x, y)$ for any $\tilde{K}$ in the range $1, \ldots, K - 1$. I show then that the result is true for K number of decisions. Let's pick four arbitrary points x', x'', y', y'' such that $x'' > x'$ and $y'' > y'$. I need to show that $u(x'', y'') + u(x', y') \geq u(x'', y') + u(x', y'')$. Let's consider a square with the nodes at the points

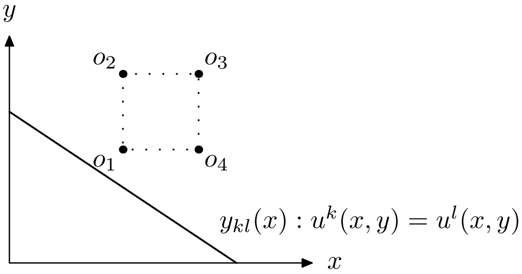
$$o_1 = (x', y'), o_2 = (x', y''), o_3 = (x'', y''), o_4 = (x'', y')$$

Let's also consider the graph of the function $y_{kl}(x)$ for some arbitrary indices k and l. I assume, without loss of generality, that the set A^{kl} lies above the graph of the function $y_{kl}(x)$ and the set A^{lk} lies below the graph of the function $y_{kl}(x)$. Let $u_{-k}(x, y)$ denote the surplus function in the case that decision k is not available for a matched pair.

$$u_{-k}(x, y) = \max_{\tilde{k} \neq k} u^{\tilde{k}}(x, y)$$

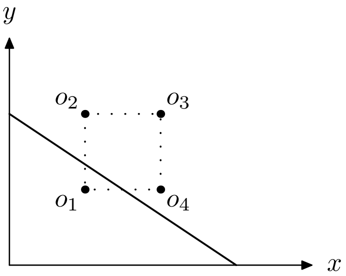
If the function $y_{kl}(x)$ is nonincreasing then all possible intersections of the graph and the square can be summarized in the following three cases.

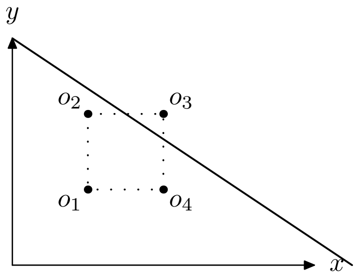
case1: the curve $y_{kl}(x)$ does not intersect the square or the curve goes through either node o_1 or node o_3. The case is shown in the picture below.



It follows from the definition of the set A^{kl} that $u^k(x,y) \geq u^l(x,y)$ at points o_1, o_2, o_3, and o_4. Therefore $u(x,y) = u_{-l}(x,y)$ at all four points. Since by induction $u_{-l}(o_1)+u_{-l}(o_3) \geq u_{-l}(o_2)+u_{-l}(o_4)$ the same inequality holds for the function $u(x,y) : u(o_1)+u(o_3) \geq u(o_2)+u(o_4)$.

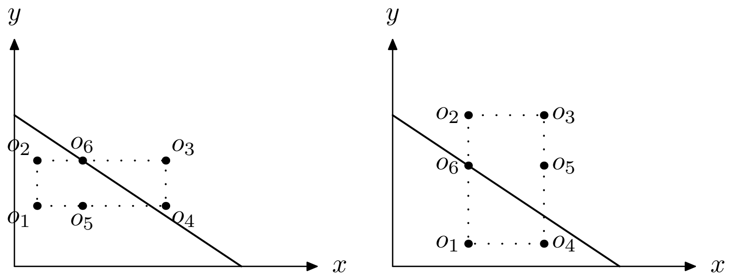
case2: the curve $y_{kl}(x)$ intersects either edges o_1o_2 and o_1o_4 or edges o_2o_3 and o_3o_4. The graph may intersect the edges at the nodes. The following pictures demonstrates the case.





Let's consider the left picture. Since $u^k(x,y) \geq u^l(x,y)$ at points o_2, o_3, and o_4 I obtain $u(x,y) = u_{-l}(x,y)$ at these three points, that is $u(o_2) = u_{-l}(o_2), u(o_3) = u_{-l}(o_3)$, and $u(o_4) = u_{-l}(o_4)$. At point o_1 I obtain $u(o_1) \geq u_{-l}(o_1)$. It follows from the inequalities that $u(o_1) + u(o_3) \geq u_{-l}(o_1) + u_{-l}(o_3) \overset{\text{by induction}}{\geq} u_{-l}(o_2) + u_{-l}(o_4) = u(o_2) + u(o_4)$. The right picture is analyzed similarly.

case 3: Let's assume that the curve $y_{kl}(x)$ either intersects the edges o_2o_3 and o_1o_4 or the edges o_1o_2 and o_3o_4. The following pictures illustrates this case.



Let's consider the left picture. Let point o_6 denote the intersection of the graph $y_{kl}(x)$ and the edge o_2o_3. The point o_5 belongs to the edge o_1o_4 and has the same x coordinate as the point o_6. In the first case I have shown that

$$u(o_1) + u(o_6) \geq u(o_2) + u(o_5) \tag{3.3.1}$$

In the second case I have shown that

$$u(o_5) + u(o_3) \geq u(o_6) + u(o_4) \tag{3.3.2}$$

Summing up the inequalities 3.3.1 and 3.3.2 I obtain

$$u(o_1) + u(o_3) \geq u(o_2) + u(o_4)$$

Similar argument applies to the right picture. This completes the proof.

■

Proposition 3.2.1 *Suppose that Assumption 3.2.1 holds. Suppose also that for each k the function $u^k(x,y)$ is supermodular and that for any indices $k > l$*

$$\frac{du^k}{dx}(x,y) \geq \frac{du^l}{dx}(x,y) \quad \text{and} \quad \frac{du^k}{dy}(x,y) \geq \frac{du^l}{dy}(x,y) \tag{3.3.3}$$

for all x and y. Then the surplus function $u(x,y)$ is supermodular and the positive assortative matching maximizes the aggregate surplus.

Proof. By definition, the function $y_{kl}(x)$ is a solution of the equation

$$u^k(x, y_{kl}(x)) = u^l(x, y_{kl}(x)) \tag{3.3.4}$$

Taking derivative of the left- and right-hand sides in 3.3.4 and rearranging the terms I obtain

$$\frac{dy_{kl}}{dx} = \frac{\frac{du^k}{dx} - \frac{du^l}{dx}}{\frac{du^l}{dy} - \frac{du^k}{dy}} \leq 0 \tag{3.3.5}$$

Therefore, the functions $y_{kl}(x)$ are nonincreasing.

■

3.3.2 Positive Assortative Matching: Necessary Conditions

Lemma 3.2.2 *Suppose that the measures μ_X and μ_Y are continuous and that the matching $y = m(x)$ maximizes the aggregate surplus. Suppose further that there exists a matched pair $\tilde{o} = (\tilde{x}, \tilde{y})$ such that $\tilde{y} = m(\tilde{x}$ and values $x^a < \tilde{x}$ and $x^b > \tilde{x}$ such that for any $x^- \in [x^a, \tilde{x})$, and $x^+ \in \left(\tilde{x}, x^b\right]$ the following inequality holds:*

$$u\left(x^-, y^-\right) + u\left(x^+, y^+\right) < u\left(x^-, y^+\right) + u\left(x^+, y^-\right) \tag{3.3.6}$$

where $y^- = m(x^-)$ and $y^+ = m(x^+)$. Then $y = m(x)$ is not positive assortative.

Proof. Suppose that matching $y = m(x)$ is positive assortative. Without loss of generality, I assume that x^a and x^b are such that $\mu_X([x^a, \tilde{x})) = \mu_X\left(\left(\tilde{x}, x^b\right]\right) > 0$. Let $y^a = m(x^a)$ and $y^b = m\left(x^b\right)$.

I show that the pairs can be rematched so that the total surplus will strictly increase. Let's consider the following matching $y = \tilde{m}(x)$. Outside the interval $\left[x^a, x^b\right]$ matching function $\tilde{m}(x)$ coincides with $m(x)$. The interval $[x^a, \tilde{x})$ is rematched away from $[y^a, \tilde{y})$ to $\left(\tilde{y}, y^b\right]$ so that matching $\tilde{m}$: $[x^a, \tilde{x}) \to \left(\tilde{y}, y^b\right]$ is positive assortative and satisfies the mass balance condition on the interval $[x^a, \tilde{x})$. Finally, I define $\tilde{m}(x)$ on the interval $\left(\tilde{x}, x^b\right]$ by the following formula: $\tilde{m}(x) = m\left(\tilde{m}^{-1}(m(x))\right)$. The matching $\tilde{m}(x)$ is shown with the dashed line in Figure 3.3.1. I know that mapping $m(x)$ satisfies the mass balance condition and that $\tilde{m}(x)$ is a one-to one mapping that satisfies mass balance condition on the interval $[x^a, \tilde{x})$. As a result, $\tilde{m}^{-1} : \left(\tilde{y}, y^b\right] \to [x^a, \tilde{x})$ is defined properly and satisfies mass balance condition. The function $\tilde{m}(x) : \left(\tilde{x}, x^b\right] \to [y^a, \tilde{y})$ also has mass balance condition property as a composition of two functions with this property.

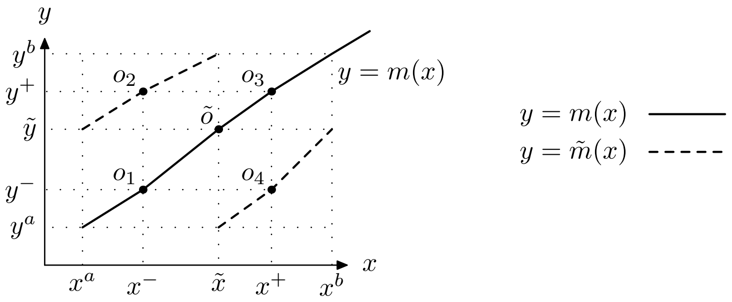


Figure 3.3.1 The positive assortative matching $m(x)$ is shown with a solid line. The alternative matching function $\tilde{m}(x)$ is shown with the dashed line. Outside the interval $\left[x^a, x^b\right]$ the two matching functions coincide.

I now show that the matching function $\tilde{m}(x)$ generates a strictly higher surplus than $m(x)$.

Let for any $x^- \in [x^a, \tilde{x})$ take $y^- = m(x^-)$, $y^+ = \tilde{m}(x^-)$, and $x^+ = m^{-1}(y^+)$. Then from the definition of $\tilde{m}(x)$ I obtain $\tilde{m}(x^+) = m(x^-) = y^-$. Therefore, from inequality 3.2.12, it follows that

$$u\left(x^-, m\left(x^-\right)\right) + u\left(x^+, m\left(x^+\right)\right) < u\left(x^-, \tilde{m}\left(x^-\right)\right) + u\left(x^+, \tilde{m}\left(x^+\right)\right)$$

Integrating the inequality from x^a to $\tilde{x}$ with respect to measure μ_X I obtain

$$\int_{x^a}^{\tilde{x}} u\left(x^-, m\left(x^-\right)\right) d\mu_X\left(x^-\right) + \int_{x^a}^{\tilde{x}} u\left(x^+, m\left(x^+\right)\right) d\mu_X\left(x^-\right) <$$
$$\int_{x^a}^{\tilde{x}} u\left(x^-, \tilde{m}\left(x^-\right)\right) d\mu_X\left(x^-\right) + \int_{x^a}^{\tilde{x}} u\left(x^+, \tilde{m}\left(x^+\right)\right) d\mu_X\left(x^-\right)$$

Next I change the variable and the limits of integration in the second terms in the left and right hand sides of the inequality. Also from the mass balance condition I obtain $d\mu_X\left(x^-\right) = d\mu_X\left(x^+\right)$. After making the changes I obtain the following inequality.

$$\int_{x^a}^{\tilde{x}} u\left(x^-, m\left(x^-\right)\right) d\mu_X\left(x^-\right) + \int_{\tilde{x}}^{x^b} u\left(x^+, m\left(x^+\right)\right) d\mu_X\left(x^+\right) <$$
$$< \int_{x^a}^{\tilde{x}} u\left(x^-, \tilde{m}\left(x^-\right)\right) d\mu_X\left(x^-\right) + \int_{\tilde{x}}^{x^b} u\left(x^+, \tilde{m}\left(x^+\right)\right) d\mu_X\left(x^+\right)$$

that is, a strictly higher total surplus is generated with matching function $\tilde{m}(x)$ than $m(x)$ and therefore $m(x)$ is not maximizing the aggregate surplus.
■

Proposition 3.2.2 *Suppose that Assumption 3.2.2 holds and that the functions $y_{kl}(x)$ are increasing functions of x. Suppose also that the measures μ_X and μ_Y are continuous. Then positive assortative matching $y = m(x)$ maximizes the aggregate surplus only if a common decision is made by all the matched pairs.*

Proof. Suppose that the matching is positive assortative but not all matched pairs make a common decision. Then there exists a point $\tilde{o} = (\tilde{x}, \tilde{y})$ such that the matched pairs with types $x < \tilde{x}$ and $y < \tilde{y}$ make decision k and matched pairs with types $x > \tilde{x}$ and $y > \tilde{y}$ make decision $l \neq k$ in some neighborhood of the point $\tilde{o}$. The point $\tilde{o}$ must lie at the intersection of the curves $y = m(x)$ and $y = y_{kl}(x)$. Figures 3.3.2 and 3.3.3 illustrate the two general possibilities for the positive assortative matching in the neighborhood of the point $\tilde{o}$.

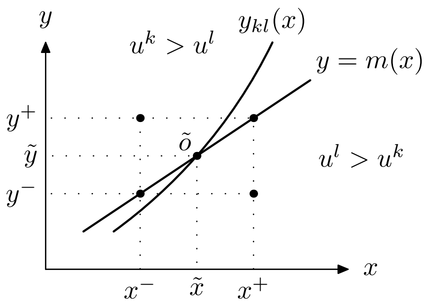


Figure 3.3.2 The matching function $y = m(x)$ intersects the line $y = y_{kl}(x)$ from above.

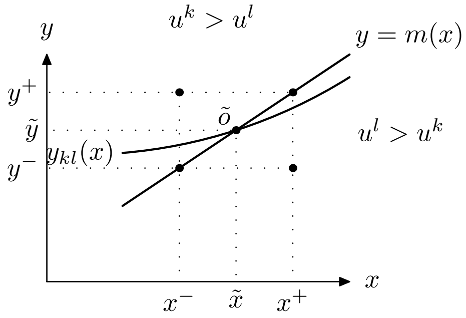


Figure 3.3.3 The matching function $y = m(x)$ intersects the line $y = y_{kl}(x)$ from below

From Figure 3.3.2 we can see that the following inequality holds

$$\frac{\partial u^k}{\partial y}(\tilde{x}, \tilde{y}) \geq \frac{\partial u^l}{\partial y}(\tilde{x}, \tilde{y}) \tag{3.3.7}$$

Since $(\tilde{x}, \tilde{y})$ lies on the graph $y_{kl}(x)$ I obtain $u^k(\tilde{x}, \tilde{y}) = u^l(\tilde{x}, \tilde{y})$. Therefore, by Assumption 3.2.2, the inequality 3.3.7 is strict. Since both functions $u^k(x, y)$ and $u^l(x, y)$ are continuously differentiable I can choose a small enough neighborhood $N_{\tilde{x}}$ of the point $\tilde{x}$ so that the following inequality holds for any points $x^- < \tilde{x}$ and $x^+ > \tilde{x}$ that belong to $N_{\tilde{x}}$.

$$\frac{u^k(x^-, y^+) - u^k(x^-, y^-)}{y^+ - y^-} > \frac{u^l(x^+, y^+) - u^l(x^+, y^-)}{y^+ - y^-}$$

where $y^- = m(x^-)$ and $y^+ = m(x^+)$. Rearranging the terms I obtain

$$u^k\left(x^-, y^-\right) + u^l\left(x^+, y^+\right) < u^k\left(x^-, y^+\right) + u^l\left(x^+, y^-\right)$$

The inequality can be rewritten as

$$u\left(x^-, y^-\right) + u\left(x^+, y^+\right) < u\left(x^-, y^+\right) + u\left(x^+, y^-\right)$$

Therefore, by Lemma 3.2.2, the positive assortative matching function $y = m(x)$ does not maximize the aggregate surplus.

Analogous arguments can be used to show that positive assortative matching, illustrated in Figure 3.3.3, does not maximize the aggregate surplus. ■

3.4 Stochastic Matching Model

In this chapter, I propose an application of the bilateral model $\mathcal{M}_2^{\mathcal{K}}$ introduced in chapter 2. As in the standard $\mathcal{M}_2^{\mathcal{K}}$ model, the utility to an agent depends on the match she forms and on the decision taken within the match. In this section's model, an idiosyncratic component which is both match and decision dependent is added to the agent's utility. This modification is meant to capture more realistic preferences over both matches and decisions.

After describing the model in section 3.4.1 I show that if the support of the idiosyncratic shock is unbounded then a unique equilibrium exists in this model. I also provide a method to compute the equilibrium transfers.

3.4.1 Model Set-Up

Let the set of male types be $X_1 = \{x_1^1, \ldots, x_1^I\}$, the set of female types be $X_2 = \{x_2^1, \ldots, x_2^J\}$, and the set of decisions be $\mathcal{K} = \{1, \ldots, K\}$. Note that unlike the model $\mathcal{M}_2^{\mathcal{K}}$ of chapter 2 I assume a finite number of agents' types. Let $\mu_1 = (\mu_1^1, \ldots, \mu_1^I)$ and $\mu_2 = (\mu_2^1, \ldots, \mu_2^J)$ denote the distributions of types of males and females. Whenever decision k is made in a match between a male of type x_1^i and a female of type x_2^j, the two agents in the match obtain the following utilities.

$$\widetilde{u}^k_{1,ij} = u^k_{1,ij} - p^k_{ij} + \eta^k_{1,ij} \tag{3.4.1}$$

$$\widetilde{u}^k_{2,ij} = u^k_{2,ij} + p^k_{ij} + \eta^k_{2,ij} \tag{3.4.2}$$

where $u^k_{1,ij}$ and $u^k_{2,ij}$ represent the deterministic components of the agents' utilities, p^k_{ij} is a transfer from the male to the female in the match, and $\eta^k_{1,ij}$ and $\eta^k_{2,ij}$ are the idiosyncratic components of the agents' utilities. I assume that $\eta^k_{1,ij}$ and $\eta^k_{2,ij}$ have continuously differentiable cumulative distribution functions, denoted by $F_{1,ij}(z)$ and $F_{2,ij}(z)$. Moreover, the random variables $\eta^k_{g,ij}$ are independent for $g = 1, 2$, $i \in \{1, ..., I\}$, $j \in \{1, ..., J\}$, and $k \in \{1, ..., K\}$. Unmatched males of type x^i_1 obtain utility $r_{1,i}$ and unmatched females of type x^j_2 obtain utility $r_{2,j}$.

I denote the $I \times J$ matrix of transfers $\left\{p^k_{ij}\right\}$ as p. Let $\tau^k_{1,ij}(p)$ denote the probability that the utility of a male of type x^i_1 attains its maximum value when he is matched with a female of type x^j_2 and decision k is made in the match. Analogously, let $\tau^k_{2,ij}(p)$ denote the probability that the utility of a female of type x^j_2 attains its maximum value when she is matched with a male of type x^i_1 and decision k is made in the match. In addition, let $\tau^0_{1,i}(p)$ denote the probability that a male of type x^i_1 best option is to stay unmatched. Probability $\tau^0_{2,j}(p)$ is defined similarly for female agents. The equilibrium of the model is defined as follows.

Definition 3.4.1 *A matrix of transfers p is an equilibrium if for all i, j, and k*

$$\mu^i_1 \tau^k_{1,ij}(p) = \mu^j_2 \tau^k_{2,ij}(p) \tag{3.4.3}$$

The above notion of equilibrium is analogous to Definition 2.6.1 in chapter 2. Given the transfers p, $\mu^i_1 \tau^k_{1,ij}(p)$ is the measure of males of type i that demand to match with females of type j and to take decision k within the match. Similarly, $\mu_2^j \tau_{2,ij}^k(p)$ is the measure of females of type j that demand to match with males of type i and to take decision k within the match. The usual mass balance condition requires that, in equilibrium, these two measures are equal. Note that the requirement that every agent makes an optimal choice is already embedded in the definition of $\tau_{g,ij}^k(p)$.

3.4.2 Existence and Uniqueness of the Equilibrium

In this section, I derive sufficient conditions for the existence and uniqueness of the equilibrium. To do so, in the next two lemmas I establish some useful properties of the probabilities $\tau_{g,ij}^k(p)$. The proofs of both results are in the Appendix.

Lemma 3.4.1 *For each i and for any $(\widetilde{j}, \widetilde{k}) \neq (j, k)$*

$$\frac{d\tau_{1,ij}^k(p)}{dp_{i\widetilde{j}}^{\widetilde{k}}} = \frac{d\tau_{1,i\widetilde{j}}^{\widetilde{k}}(p)}{dp_{ij}^k} \tag{3.4.4}$$

Analogously, for each j and for any $(\widetilde{i}, \widetilde{k}) \neq (i, k)$

$$\frac{d\tau_{2,ij}^k(p)}{dp_{\widetilde{i},j}^{\widetilde{k}}} = \frac{d\tau_{2,\widetilde{i}j}^{\widetilde{k}}(p)}{dp_{ij}^k} \tag{3.4.5}$$

Lemma 3.4.2 *The functions $\tau_{1,ij}^k(p)$ and $\tau_{2,ij}^k(p)$ have the following properties:*

1. *$\tau_{1,ij}^k(p)$ is decreasing in p_{ij}^k ($\tau_{2,ij}^k(p)$ is increasing in p_{ij}^k)*
2. *$\tau_{1,i}^0(p)$ is increasing in p_{ij}^k ($\tau_{2,j}^0(p)$ is decreasing in p_{ij}^k)*
3. *$\tau_{1,ij}^k(p)$ is increasing in $p_{i\widetilde{j}}^{\widetilde{k}}$ for any $(\widetilde{j}, \widetilde{k}) \neq (j, k)$ ($\tau_{2,ij}^k(p)$ is decreasing in $p_{\widetilde{i}j}^{\widetilde{k}}$ for any $(\widetilde{i}, \widetilde{k}) \neq (i, k)$)*

4. $\sum_{\widetilde{j},\widetilde{k}} \frac{d\tau^k_{1,ij}(p)}{dp^{\widetilde{k}}_{i\widetilde{j}}} \leq 0$ $\quad (\sum_{\widetilde{i},\widetilde{k}} \frac{d\tau^k_{2,ij}(p)}{dp^{\widetilde{k}}_{\widetilde{i}j}} \geq 0)$. *Moreover, if* $r^i_1 > -\infty$ *and* $r^j_2 > -\infty$ *for any* i *and* j*, and for any* i, j, k *and* g *support of* $\eta^k_{g,ij}$ *is* $(-\infty, \infty)$*, then the inequalities are strict.*

In the Appendix, using Lemmas 3.4.1 and 3.4.2 I provide a proof by construction of the main result that an equilibrium exists and is unique in this model.

Theorem 3.4.1 *Suppose that, for any* g, i, j *and* k*, the distributions of* $\eta^k_{g,ij}$ *are continuous on the support* $(-\infty, \infty)$*. Also suppose that* $r^i_1 > -\infty$ *and* $r^j_2 > -\infty$ *for any* i *and* j*. There exists a unique equilibrium of the model. Moreover, the equilibrium can be constructed as a limit of the sequence*

$$p^{(n+1)} = p^{(n)} + \lambda \left[\mu^i_1 \tau^k_{1,ij}(p^{(n)}) - \mu^j_2 \tau^k_{2,ij}(p^{(n)})\right] \qquad n = 1, 2, \ldots \qquad (3.4.6)$$

for some sufficiently small $\lambda > 0$*.*

Beyond establishing existence and uniqueness of equilibrium Theorem 3.4.1 also provides a simple method for computing the equilibrium transfers.

3.4.3 Appendix

Next, I formulate some results that complete the proof of the existence and uniqueness of the equilibrium of the general model in this chapter.

Existence and Uniqueness of the Solution of a System of Nonlinear Equations

There is an extensive literature on this subject some of which, [26], is mentioned in the references. However, none of the results can be applied directly in this case. Thus, I need to manipulate the proofs in the literature. Below I provide conditions under which (i) there exists a unique solution of the system and (ii) the solution can be obtained as a fixed point of a properly defined contraction mapping.

The notation, that I use in this subsection, is independent from the notation in the rest of this paper. The results in this subsection are general and they are the reformulation of standard results to be applied in this particular case.

Theorem 3.4.2 *Suppose that a vector function $h_i(p_1, \ldots, p_n)$, $i = 1 \ldots n$ is continuously differentiable for each i and has the following properties.*

1. *For any i, j and any p*

$$\frac{dh_i}{dp_i} < 0, \qquad \frac{dh_i}{dp_j} > 0 \text{ for } j \neq i, \qquad \left|\frac{dh_i}{dp_i}\right| > \sum_{j \neq i} \frac{dh_i}{dp_j} \tag{3.4.7}$$

2. *There exists a constant $A > 0$ such that for and any vector p such that $\max_i \mid p_i \mid = A$ there exists index j such that*

$$\begin{array}{rl} \text{either} & h_j(p) < 0 \text{ and } p_j > 0 \\ \text{or} & h_j(p) > 0 \text{ and } p_j < 0 \end{array} \tag{3.4.8}$$

Then there exists a unique solution of the system of equations

$$h_i(p_1, \ldots, p_n) = 0, \quad i = 1, \ldots, n \tag{3.4.9}$$

Note: condition $\frac{dh_i}{dp_j} > 0$ for $j \neq i$ is not essential but it simplifies the proof of the theorem.

Let define set S as

$$S = \{p : |p_i| \leq A \quad \text{for all } i\} \tag{3.4.10}$$

and consider the following metric ρ on $\mathbb{R}^n$

$$\rho(p^{(1)}, p^{(2)}) = \max_i \left| p_i^{(1)} - p_i^{(2)} \right| \tag{3.4.11}$$

The theorem follows from the following proposition.

Proposition 3.4.1 *The transformation* $\mathcal{T}(p) = (\mathcal{T}_1(p), \ldots, \mathcal{T}_n(p)$ *defined as*

$$\mathcal{T}_i(p) = \begin{cases} p_i + \lambda h_i(p) & \text{if} \quad -A \leq \; p + \lambda h_i(p) \; \leq A \\ -A & \text{if} \qquad\qquad p + \lambda h_i(p) \; \leq -A \\ A & \text{if} \qquad A \leq \; p + \lambda h_i(p) \end{cases} \tag{3.4.12}$$

is a contraction mapping of the metric space (S, ρ) *into itself for some* $\lambda > 0$.

Proof. By construction, $\mathcal{T}$ maps S into itself. Transformation $\mathcal{T}$ is a composition, $\mathcal{T} = \mathcal{T}^1 \circ \mathcal{T}^2$, of two transformations:

$$\mathcal{T}_i^2(p) = p_i + \lambda h_i(p) \tag{3.4.13}$$

and $\mathcal{T}^1$ is a projection on set S, $\mathcal{T}_i^1(p) = \max(-A, \min(A, p_i))$. From the definition of $\mathcal{T}^1$ it follows directly that $\rho(\mathcal{T}^1(p^{(1)}), \mathcal{T}^1(p^{(2)})) \leq \rho(p^{(1)}, p^{(2)})$. Therefore, to show that $\mathcal{T}$ is a contraction mapping, it is sufficient to show that $\mathcal{T}^2$ is a contraction mapping.

To show that $\mathcal{T}^2$ is a contraction mapping, I need to show that there exists $0 < \alpha < 1$ such that for any $p^{(1)} \in S$ and $p^{(2)} \in S$ the following inequality holds: $\rho(\mathcal{T}^2(p^{(1)}), \mathcal{T}^2(p^{(2)})) \leq \alpha \rho(p^{(1)}, p^{(2)})$. Consider

$$\alpha = 1 - \lambda \min_{i, \widetilde{p} \in S} \left[\left| \frac{dh_i}{dp_i}(\widetilde{p}) \right| - \sum_{j \neq i} \left| \frac{dh_i}{dp_j}(\widetilde{p}) \right| \right] \tag{3.4.14}$$

Since $\left| \frac{dh_i}{dp_i}(\widetilde{p}) \right| - \sum_{j \neq i} \left| \frac{dh_i}{dp_j}(\widetilde{p}) \right| > 0$ for any $\widetilde{p} \in S$ and set S is a compact I obtain $\alpha < 1$. Since $\frac{dh_i}{dp_i} < 0$ and $\frac{dh_i}{dp_j} \geq 0$ for $j \neq i$ I can rewrite 3.4.14

as $\alpha = 1 + \lambda \max_{i,\widetilde{p}\in S} \sum_j \frac{dh_i}{dp_j}(\widetilde{p})$. I can also choose λ small enough so that $1 + \lambda \max_{\widetilde{p}\in S} \frac{dh_i}{dp_i}(\widetilde{p}) > 0$.

Pick some arbitrary points $p^{(1)} \in S$ and $p^{(2)} \in S$. For any i there exists a point[3] $\widetilde{p} = tp^{(1)} + (1-t)p^{(2)}$ for some $t \in [0,1]$ such that

$$T_i^2(p^{(2)}) - T_i^2(p^{(1)}) = \left(p_i^{(2)} - p_i^{(1)}\right) + \lambda \sum_j \frac{dh_i(\widetilde{p})}{dp_j} \left(p_j^{(2)} - p_j^{(1)}\right) \qquad (3.4.15)$$

which can be rewritten as

$$\left|T_i^2(p^{(2)}) - T_i^2(p^{(1)})\right| = \left|\left(1 + \lambda \frac{dh_i(\widetilde{p})}{dp_i}\right)\left(p_i^{(2)} - p_i^{(1)}\right) + \lambda \sum_{j\neq i} \frac{dh_i(\widetilde{p})}{dp_j}\left(p_j^{(2)} - p_j^{(1)}\right)\right|$$
$$\leq \left|\left(1 + \lambda \frac{dh_i(\widetilde{p})}{dp_i} + \lambda \sum_{j\neq i} \frac{dh_i(\widetilde{p})}{dp_j}\right)\right| \rho(p^{(1)}, p^{(2)}) \leq \alpha \rho(p^{(1)}, p^{(2)}) \qquad (3.4.16)$$

and this proves the proposition. ■

The proposition shows that $\mathcal{T}(p)$ is a contraction mapping on (S, ρ). Therefore, there exists a unique fixed point $\widehat{p}$ of the transformation $\mathcal{T}$. That is, there exists a unique point in S such that $\mathcal{T}(\widehat{p}) = \widehat{p}$. This, however, does not prove Theorem 3.4.2 yet. The fixed point $\widehat{p}$ of the transformation $\mathcal{T}$ can belong to the boundary of the set S in which case it may not be the fixed point of the transformation $\mathcal{T}^1$. To prove the theorem it is sufficient to show that if A is such that condition 3.4.8 of Theorem 3.4.2 holds, then the fixed point of $\mathcal{T}$ does not belong to the boundary of the set S.

Proof (of theorem 3.4.2). To prove the theorem it is sufficient to show that there does not exist $\widehat{p}$ that belongs to the boundary of S and $\mathcal{T}(\widehat{p}) = \widehat{p}$. Let $\widehat{p}$ be some arbitrary point on the boundary of S. From the second condition of the theorem it follows that either there exists j such that $h_j(p) < 0$ and $p_j > 0$ or there exists j such that $h_j(p) > 0$ and $p_j < 0$. Let's

[3]This statement is a simple extension of a standard result from calculus.

assume that $h_j(p) < 0$ and $p_j > 0$ for some j (the other case is considered analogously). Since $\mathcal{T}_j^2(p) = p_j + \lambda h_j(p) > p_j$, we obtain $\mathcal{T}_j(p) = \mathcal{T}_j^1 \circ \mathcal{T}_j^2(p) = \min(A, p_j + \lambda h_j(p)) > p_j$. Therefore, $\mathcal{T}(p) \neq p$. This proves the theorem. ■

Existence and Uniqueness of Equilibrium

Lemma 3.4.1 *For each i and for any $(\widetilde{j}, \widetilde{k}) \neq (j, k)$*

$$\frac{d\tau^k_{1,ij}(p)}{dp^{\widetilde{k}}_{i\widetilde{j}}} = \frac{d\tau^{\widetilde{k}}_{1,i\widetilde{j}}(p)}{dp^k_{ij}} \tag{3.4.17}$$

Analogously, for each j and for any $(\widetilde{i}, \widetilde{k}) \neq (i, k)$

$$\frac{d\tau^k_{2,ij}(p)}{dp^{\widetilde{k}}_{\widetilde{i},j}} = \frac{d\tau^{\widetilde{k}}_{2,\widetilde{i}j}(p)}{dp^k_{ij}} \tag{3.4.18}$$

Proof. I prove only equation 3.4.17. Equation 3.4.18 can be proved analogously. I remind that

$$\tau^k_{1,ij} = \Pr\left(\widetilde{u}^k_{1,ij} \geq \max_{(j',k') \neq (j,k)} \widetilde{u}^{k'}_{1,ij'}\right) \tag{3.4.19}$$

and

$$\widetilde{u}^k_{1,ij} = u^k_{1,ij} - p^k_{ij} + \eta^k_{1,ij} \tag{3.4.20}$$

Let's fix the values of $u^{\widetilde{\widetilde{k}}}_{i\widetilde{\widetilde{j}}}$ for $(\widetilde{\widetilde{j}}, \widetilde{\widetilde{k}}) \neq (j, k)$ and $(\widetilde{\widetilde{j}}, \widetilde{\widetilde{k}}) \neq (\widetilde{j}, \widetilde{k})$ and let's denote

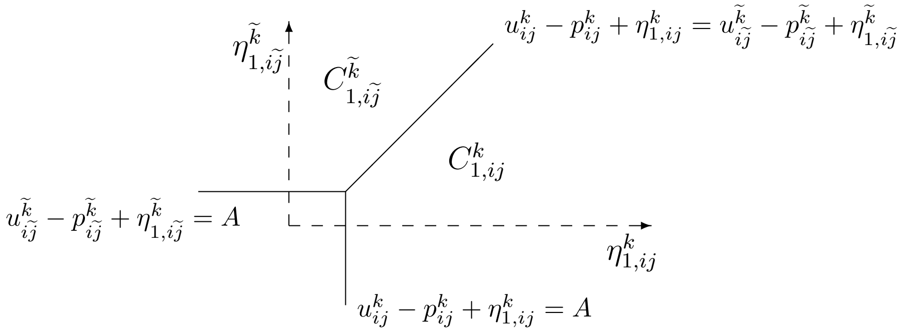
$$A = \max_{\substack{(\widetilde{\widetilde{j}}, \widetilde{\widetilde{k}}) \neq (j,k) \\ (\widetilde{\widetilde{j}}, \widetilde{\widetilde{k}}) \neq (\widetilde{j}, \widetilde{k})}} u^{\widetilde{\widetilde{k}}}_{i\widetilde{\widetilde{j}}} \tag{3.4.21}$$

Let's consider the following sets.

$$C^k_{1,ij} = \left\{(\eta^k_{1,ij}, \eta^{\widetilde{k}}_{1,i\widetilde{j}}) : \widetilde{u}^k_{1,ij} \geq \widetilde{u}^{\widetilde{k}}_{1,i\widetilde{j}} \quad \text{and} \quad \widetilde{u}^k_{1,ij} \geq A\right\} \tag{3.4.22}$$

$$C^{\widetilde{k}}_{1,i\widetilde{j}} = \left\{(\eta^k_{1,ij}, \eta^{\widetilde{k}}_{1,i\widetilde{j}}) : \widetilde{u}^{\widetilde{k}}_{1,i\widetilde{j}} \geq \widetilde{u}^k_{1,ij} \quad \text{and} \quad \widetilde{u}^{\widetilde{k}}_{1,i\widetilde{j}} \geq A\right\} \tag{3.4.23}$$

The sets $C^k_{1,ij}$ and $C^{\widetilde{k}}_{1,i\widetilde{j}}$ are illustrated in the figure below.



Let's now compare the change in the set $C^{\tilde{k}}_{1,i\tilde{j}}$ as p^k_{ij} is increased by some small ϵ (Figure 3.4.1) with the change in the set $C^k_{1,ij}$ as $p^{\tilde{k}}_{i\tilde{j}}$ is decreased by ϵ (Figure 3.4.2).

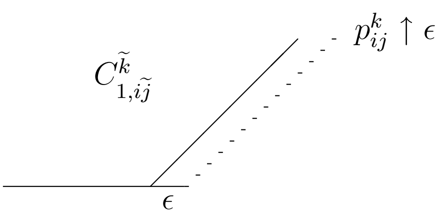


Figure 3.4.1 *The left panel shows expansion of the set $C^{\tilde{k}}_{1,i\tilde{j}}$ as p^k_{ij} increases by ϵ.*

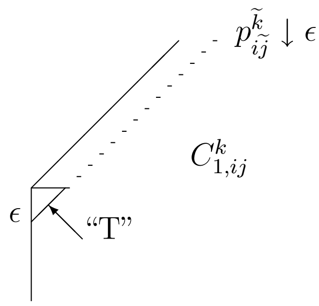


Figure 3.4.2 *The right panel shows contraction of the set $C^k_{1,ij}$ as $p^{\tilde{k}}_{i\tilde{j}}$ decreases by ϵ.*

By definition,

$$\tau^k_{1,ij} = \Pr\left(C^k_{1,ij}\right) \qquad \text{and} \qquad \tau^{\tilde{k}}_{1,i\tilde{j}} = \Pr\left(C^{\tilde{k}}_{1,i\tilde{j}}\right) \tag{3.4.24}$$

As I have shown, the change in the sets $C^k_{1,ij}$ and $C^{\tilde{k}}_{1,i\tilde{j}}$, that corresponds to the change in $p^{\tilde{k}}_{i\tilde{j}}$ and $p^k_{1,ij}$, is the same except the triangle "T" shown in Figure 3.4.2. But the measure of the triangle "T" is of ϵ^2 order and can be ignored. This proves that conditional on the vector of random variables

$\left(u^{\widetilde{\widetilde{k}}}_{i\widetilde{\widetilde{j}}}\right)_{(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(j,k),\,(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(\widetilde{j},\widetilde{k})}$ the derivative of $\tau^{k}_{1,ij}$ with respect to $p^{\widetilde{k}}_{i\widetilde{j}}$ is equal to the derivative of $\tau^{\widetilde{k}}_{1,i\widetilde{j}}$ with respect to $p^{k}_{1,ij}$.

$$\frac{d\Pr\left(C^{k}_{g,ij}\middle|\left(u^{\widetilde{\widetilde{k}}}_{i\widetilde{\widetilde{j}}}\right)_{(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(j,k),\,(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(\widetilde{j},\widetilde{k})}\right)}{dp^{\widetilde{k}}_{i\widetilde{j}}} = \frac{d\Pr\left(C^{\widetilde{k}}_{g,i\widetilde{j}}\middle|\left(u^{\widetilde{\widetilde{k}}}_{i\widetilde{\widetilde{j}}}\right)_{(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(j,k),\,(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(\widetilde{j},\widetilde{k})}\right)}{dp^{k}_{ij}} \qquad (3.4.25)$$

Taking expectation of 3.4.25 with respect to $u^{\widetilde{\widetilde{k}}}_{i\widetilde{\widetilde{j}}}$ for all $(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(j,k)$ and $(\widetilde{\widetilde{j}},\widetilde{\widetilde{k}})\neq(\widetilde{j},\widetilde{k})$ I obtain 3.4.17. This proves the lemma. ■

Proposition 3.4.2 *The functions $\tau^{k}_{1,ij}(p)$ and $\tau^{k}_{2,ij}(p)$ have the following properties:*

Proof. The proof of Lemma 3.4.1 can be used to show the first three properties. The only nontrivial property[4] is the last one, $\sum_{\widetilde{j},\widetilde{k}}\frac{d\tau^{k}_{1,ij}}{dp^{\widetilde{k}}_{i\widetilde{j}}}\leq 0$. The inequality can be proved by the following argument. By definition,

$$\sum_{\widetilde{j},\widetilde{k}}\tau^{\widetilde{k}}_{1,i\widetilde{j}}(p)+\tau^{0}_{1,i}(p)\equiv 1 \qquad (3.4.26)$$

for any p. Taking the derivative of 3.4.26 with respect to p^{k}_{ij} I obtain

$$\sum_{\widetilde{j},\widetilde{k}}\frac{\tau^{\widetilde{k}}_{1,i\widetilde{j}}(p)}{dp^{k}_{ij}}+\frac{\tau^{0}_{1,i}(p)}{dp^{k}_{ij}}=0 \qquad (3.4.27)$$

But by Lemma 3.4.1

[4]the property $\sum_{\widetilde{i},\widetilde{k}}\frac{d\tau^{k}_{2,ij}}{dp^{\widetilde{k}}_{\widetilde{i}j}}\geq 0$ can be proved by the same argument

$$\frac{d\tau^{\widetilde{k}}_{g,i\widetilde{j}}(p)}{dp^{k}_{ij}} = \frac{d\tau^{k}_{g,ij}}{dp^{\widetilde{k}}_{i\widetilde{j}}(p)} \tag{3.4.28}$$

and, therefore, substituting 3.4.28 into 3.4.27 I obtain

$$\sum_{\widetilde{j},\widetilde{k}} \frac{\tau^{k}_{1,ij}}{dp^{\widetilde{k}}_{i\widetilde{j}}} + \frac{\tau^{0}_{1,i}(p)}{dp^{k}_{ij}} = 0 \tag{3.4.29}$$

and inequality $\sum_{\widetilde{j},\widetilde{k}} \frac{\tau^{k}_{1,ij}}{dp^{\widetilde{k}}_{i\widetilde{j}}} \leq 0$ follows from the inequality $\frac{\tau^{0}_{1,i}(p)}{dp^{k}_{ij}} \geq 0$. Moreover, if for any i, j and k support of $\eta^{k}_{1,ij}$ is $(-\infty, \infty)$, then $\frac{\tau^{0}_{1,i}(p)}{dp^{k}_{ij}} > 0$. Therefore, $\sum_{\widetilde{j},\widetilde{k}} \frac{\tau^{k}_{1,ij}}{dp^{\widetilde{k}}_{i\widetilde{j}}} < 0$. ■

Chapter 4

Matching with Coordination Frictions

4.1 Introduction

A variety of models has been proposed in the literature that analyze the matches between workers and jobs that take place in the labor market. The first examples of models which are still used as a standard framework to describe the labor market are given in [32] (Shapley and Shubik, 1972) and in [2] (Becker, 1973). In these papers, the labor market is modelled as an assignment model and the solution of the assignment model is the equilibrium in the labor market. The model assumes no frictions in the economy. Each worker can apply to any job in the economy and there is a centralized mechanism that assigns workers to jobs.

There are many reasons to believe that these assumptions are not realistic in practice. The predictions of this frictionless economy that are inconsistent with empirical evidence are discussed in many papers that study a model with some *coordination frictions* in the economy (like [33] (Shi, 2001) or [34],[35] (Shimer, 1996, 2001). Coordination frictions may result in an inefficient allocation of workers to firms. That is, in the presence of frictions, there may be too many applicants at some firms and too few at others. As a result, some workers are not matched and some vacancies are not filled. In the papers cited above and in the model of this chapter, there are three different sources of coordination frictions. First, each worker can apply only to a single firm in the economy. Therefore, if a worker does not obtain a job at the firm to which he applied, he stays unmatched as he can not apply to a different firm. Second, a worker does not observe the types of other workers and, therefore, can not predict the choices of other workers perfectly. As a result, a firm can receive too many or too few applications and, ex-post, some workers would be better off if they had applied to a different firm. Finally, it is assumed that workers use symmetric strategies so that each worker expects that all workers of a given type choose a common application strategy. Since workers can not coordinate their actions there may be too many applications to some firms and too few to others.

The idea of frictions can be illustrated by the following simple example. Let's assume that there are two identical workers, two identical firms, and each firm has a single available job vacancy (since workers are identical, they know each other type and, therefore, there are no frictions related to imperfect information). Each worker can apply only to a single firm. A coordination mechanism would assign worker one to firm one and worker two to firm two. In the absence of such coordination, however, worker one does not know where worker two is applying. Assuming that both workers choose a common uncoordinated strategy, it can be shown that the equilibrium outcome is that each worker applies with probability $\frac{1}{2}$ to each firm. Therefore there is a 50% chance that both workers would apply to one and the same firm.

Interest in matching models with coordination frictions has been largely motivated by the fact that the models seem to do a better job than those without frictions in describing the facts that are observed in the labor market. The literature began with the matching models in which workers and firms are homogeneous [24] (Montgomery, 1991), [7] (Burdett, Shi, and Wright, 2001). These models have been extended to include heterogeneous one-dimensional types of firms and workers [35] (Shimer, 2001).

However, the assumption that the types of workers and firms are one-dimensional makes it impossible in some cases to estimate the labor market quantitatively. For example, if you model the flows of workers from one geographic area to another, the assumption of one dimensional types would be unrealistic (The types must include, at a minimum, the current geographic location of a worker or a firm). In general, depending on questions being asked, the types of workers may need to include such characteristics as age, education, current geographical location, sex, marital status, or residence status. Likewise, the firms' types may include a variety of characteristics like physical capital or geographical location.

I propose a model of a *static labor market with coordination frictions, a finite number of types available for each agent, a general form of heterogeneity of firms and workers, and a finite number of job positions at each firm.*[1] I not only assume a general form of heterogeneity of the agents' types but I also allow *more than one* job position at each firm. While a worker's productivity is independent of the job the firm's reserve value for a job depends on the job. The size of the firms is determined endogenously in the model.[2]

Methodologically, my approach to the problem differs from that in the

[1] I consider separately two cases, one with a finite number of workers and firms of each type and the other with a continuum of each.

[2] Because I assume that the job positions are homogeneous in each firm, the model is applied only to markets for a specific specialty, like accounting, etc.

previous literature. In the previous literature, worker types can be ordered in such a way that lower types are less productive than higher types. This allows authors to obtain an equilibrium of the Bayesian game analytically. However, worker types in my model are not assumed to be one dimensional so that worker types may not be ordered. There is no direct analytic solution. Instead, I focus on the planner's optimization problem associated with the model and derive its properties. I show that the solution of the optimization problem can be interpreted as the symmetric equilibrium of a Bayesian game that models the behavior in the labor market. In the cited literature the focus is on the equilibrium of the model which is shown to coincide with the planner's solution. There are two reasons why I solve the model indirectly by focusing on the planner's optimization problem instead of the equilibrium of the model.

1. The planner's problem is *tractable*. I give conditions under which the planner's problem is a concave optimization problem. Although there is no direct analytical solution, this fact allows me to propose a new use for standard algorithms in constructing a solution to the planner's problem.

2. This approach offers a natural way to construct symmetric equilibria in a *class of Bayesian games* for which there are no direct analytical solutions. The class of Bayesian games that I describe in section 4.3 has other than labor market applications. For example, in section 4.4, I present a game with imperfect information in which firms choose simultaneously different markets. If more than one firm enter the same market the firms compete in Bertrand in this market. The paper shows how to construct symmetric equilibria in these types of games.

The approach also allows me to show that certain results of the previous literature are robust to the extension of the model that I propose. In particular,

- the equilibrium in the market is constrained efficient (that is, it coincides with the solution of the constrained planner's optimization problem).
- Both competition and wages increase in the labor market as the number of firms increases.[3]

The paper is organized as follows. In section 4.2, I introduce the model, denoted by $\mathcal{M}^I$, with coordination frictions, with a finite number of agents of each type and an arbitrary but finite number of jobs at each firm. I show that the planner's optimization problem, associated with the model, is a concave maximization problem.

In section 4.3, I show that $\mathcal{M}^I$ can be represented as a Bayesian game. The solution of the optimization problem associated with $\mathcal{M}^I$ coincides with the symmetric equilibrium of the Bayesian game.

To formulate the model with coordination frictions in the case of a continuum number of agents, I introduce first in section 4.5 a sequence of $\mathcal{M}^I_n$ models with a special choice of parameters of the models. The model $\mathcal{M}^\infty$ with a continuum number of agents is interpreted as a limit of the sequence $\mathcal{M}^I_n$. A formal description of $\mathcal{M}^\infty$ and its properties is given in section 4.6. In particular, in section 4.6.4, I show how the optimal flows of workers to firms can be constructed numerically in the case that there is a single job position in each firm.

Finally, section 4.7 concludes the paper.

[3]The wages decrease as the number of workers increases.

4.2 Discrete Model

In this section, I describe a matching model in which there are coordination frictions, a finite number of types of each agent, a finite number of workers and firms, and an arbitrary but finite number of jobs at each firm. I denote the model by $\mathcal{M}^I$.

4.2.1 Model Set Up

$\mathcal{M}^I$ is described as follows.

Agents

There are $i = 1, \ldots, I$ workers and $m = 1, \ldots, M$ firms. Let K_m denote the number of job positions available at firm m and let r_{jm} denote the firm's reserve value of job $j = 1, \ldots, K_m$ at firm m. Firm m with job position j fills a job with reserve value r_{jm} only if the worker's productivity at this firm is greater than or equal to r_{jm}.

Preferences

Both workers and firms are risk-neutral so that workers maximize expected wages and firms maximize expected profits. Utility of the agents is transferrable. The reservation utility of each worker is zero.

Time and uncertainty

There are two periods of time: $t = 0$ (*ex-ante* stage) and $t = 1$ (*ex-post* stage). At $t = 0$, every worker is identical. At $t = 1$, each worker learns his type $g \in \{1, \ldots, G\}$. A worker's type is randomly and *independently* drawn according to a *common* distribution $f = (f_1, \ldots, f_G)$. Let ω_{gm} denote the productivity of a worker of type g at firm m. I note that it is possible for $\omega_{gm} < \omega_{g'm}$ even though $\omega_{gm'} > \omega_{g'm'}$. Thus, one may not be able to rank worker types g according to productivity as any ranking also depends on m. If there were a continuum of types, then this inability to rank worker types according to productivity would be modelled by the existence of multi-dimenional types. Since there is a finite number of types, the aspect of multi-dimensionality is captured by the fact that one cannot separate the types into higher and lower types as we cannot order the types of firms and workers in such a way that productivity increases in each variable separately.

Application probabilities

Each worker applies only to a single firm. The probability that worker i of type g applies to firm m is denoted as p^i_{gm}. Let $\mathbf{p}^i = (p_{gm})^i$ denote a $G \times M$ matrix of application probabilities of worker i. In this paper, I consider only symmetric application probabilities

$$\mathbf{p}^i = \mathbf{p}^j \quad \text{for any } i, j \tag{4.2.1}$$

Let $\mathbf{p} = (p_{gm})$ denote a common $G \times M$ matrix of application probabilities.

Matches in a firm

Suppose that a subset $\mathcal{I}_m$ of workers of types $(g_1, \dots g_L)$ applies to firm m and the workers are ordered in such a way that their productivities at the firm are nondecreasing: $\omega_{g_1 m} \geq \omega_{g_2 m} \geq \dots \geq \omega_{g_L m}$. Suppose also that the reserve values r_{jm} of the job vacancies at firm m are ordered in nonincreasing order: $r_{1,m} \leq r_{2,m} \leq \dots \leq r_{K_m,m}$. I assume that the most productive worker from the set $\mathcal{I}_m$ is matched with the job position that has the lowest reserve value, the worker with the second highest productivity is matched with the job position that has the second lowest reserve value, etc. The workers and the job position are matched in this way until the productivity of the next matched worker is smaller than the reserve value of the next matched job position.[4] The matching is illustrated in Figure 4.2.1.

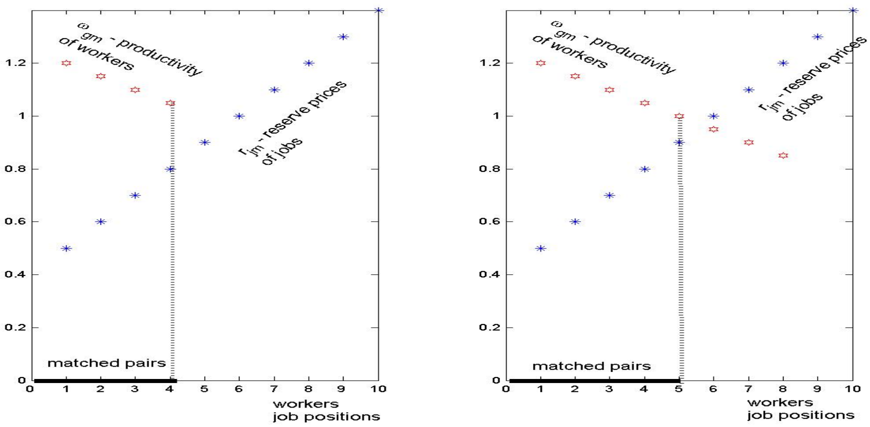


Figure 4.2.1 *The reserve values $r_{j,m}$ correspond to the supply of jobs at firm m. The productivities $\omega_{g_i m}$ of the workers from the set $\mathcal{I}_m$ correspond to the demand for job positions. On the left panel, all workers who apply to firm m are matched to jobs at the firm. On the right panel, the workers and the firm are matched up to the point where the lowest reserve value of unfilled job positions exceeds the highest productivity of unmatched worker. Some workers and jobs in the example stay unmatched.*

Surplus

Let $\omega_{gm} - r_{jm}$ denote the surplus generated in a match between a worker of type g and job j at firm m. Suppose that types of workers

[4] This is one way for a planner to choose a set of matches to maximize the aggregate surplus, generated at the firm. What matters is which set of worker types are assigned to which set of jobs at each firm and not the exact matching of specific workers to specific jobs within the two sets.

are $\mathbf{g} = (g_1, \ldots, g_I)$ and worker i of type g_i applies to firm m_i. Let $\mathbf{m} = (m_1, \ldots, m_I)$. Then the surplus generated in firm m, denoted by $\widetilde{W}_m(\mathbf{g}, \mathbf{m})$, is the sum of surpluses generated in all the matches in firm m given the set $\mathcal{I}_m$ of types $g_i \in (g_1, \ldots, g_I)$ for whom $m_i = m$. The aggregate surplus, denoted by $\widetilde{W}(\mathbf{g}, \mathbf{m})$, is a sum of surpluses generated at each firm

$$\widetilde{W}(\mathbf{g}, \mathbf{m}) = \sum_m \widetilde{W}_m(\mathbf{g}, \mathbf{m}) \tag{4.2.2}$$

Since each worker's type is randomly and *independently* drawn according to a *common* distribution $f = (f_1, \ldots, f_G)$, if each worker i uses a common $G \times M$ matrix $\mathbf{p} = (p_{gm})$ of application probabilities then, the expected surplus (denoted by $W_m(\mathbf{p})$) generated at firm m satisfies

$$W_m(\mathbf{p}) = \mathtt{E}_{\mathbf{g},\mathbf{m}} \left(\widetilde{W}_m(\mathbf{g}, \mathbf{m}) \mid \mathbf{p} \right) \tag{4.2.3}$$

where the expectation is taken with respect to realizations of $\mathbf{g}$ and $\mathbf{m}$ for a given matrix of application probabilities $\mathbf{p}$. The aggregate expected surplus (denoted by $W(\mathbf{p})$) satisfies

$$W(\mathbf{p}) = \mathtt{E}_{\mathbf{g},\mathbf{m}} \left(\widetilde{W}(\mathbf{g}, \mathbf{m}) \mid \mathbf{p} \right) \tag{4.2.4}$$

Solution concept

A solution of the model is a $G \times M$ matrix of symmetric application probabilities $\mathbf{p} = (p_{gm})$ that maximizes the expected aggregate surplus $W(\mathbf{p})$. The optimization problem is described formally in the next section. In section 4.3, I show that the model can be represented as a Bayesian game and that the solution of the optimization problem $\mathbf{p}$ can be interpreted as a symmetric equilibrium of the game.

4.2.2 Optimization Problem

The surplus maximizing application probabilities of workers to jobs are described by a $G \times M$ matrix $\mathbf{p} = (p_{gm})$ that solves the following optimization problem

$$\begin{cases} \max_p W(\mathbf{p}) \\ \sum_m p_{gm} \leq 1 \\ p_{gm} \geq 0 \end{cases} \tag{4.2.5}$$

where $W(\mathbf{p}) = \sum_m W_m(\mathbf{p})$ is the aggregate expected surplus generated by all matches in the market. Let $\mathbf{p}_m = (p_{1m}, \ldots, p_{Gm})$ denote the application probabilities of workers of types $g = 1, \ldots, G$ to firm m. Note that $W_m(\mathbf{p})$ does not depend on the application probabilities to firms other than firm m. Therefore, at the risk of abusing notation, I write the function $W_m(\mathbf{p}_m)$ as a function of the argument $\mathbf{p}_m$ only. For different restrictions on the parameters of the model, I give explicit descriptions of the function $W_m(\mathbf{p}_m)$ in section 4.8.1 in the appendix of the paper.

The following theorem provides sufficient conditions under which the objective function in 4.2.5 is concave.

Theorem 4.2.1 *Suppose that either*

(a) Firm $m \in \{1, ...M\}$ has a constant reserve value of each job position (that is, for each m, $r_{jm} = r_m$ for all j)

or

(b) The productivity of a worker does not depend on the type of the worker at firm $m \in \{1, ...M\}$ (that is, for each m, $\omega_{gm} = \omega_m$ for any g)

Then the expected surplus function $W(\mathbf{p})$ is a concave function of a $G \times M$ matrix of application probabilities $\mathbf{p}$ on the set of all feasible matrices of application probabilities.

The proof of the theorem is given in section 4.8.3 (page 147) in the the appendix of the paper. The first condition above says, from the point of view of a firm, jobs are homogeneous since the reserve value is constant across jobs at any given firm. The second condition above states that, from the point of view of any particular firm, workers are homogeneous ex post since a worker's productivity is constant across worker types at any given firm. Though a direct analytical solution to the planner's problem 4.2.5 may not exist in general, there are standard algorithms which solve the planner's problem 4.2.5 whenever the objective function is concave. Thus, whenever conditions (a) or (b) hold, one can obtain the solution to the planner's problem 4.2.5.

4.3 Matching with Frictions as a Bayesian Game

In this section, I show that the solution of the optimization problem 4.2.5 is a symmetric equilibrium of a suitably defined Bayesian game. The Bayesian game describes a decentralized labor market with coordination frictions. In the market, the workers choose firms simultaneously and each worker applies to a single firm. The frictions are modelled by the following three assumptions: (1) each worker can send an application only to a single firm, (2) each worker observes only his own type but not the type of the other workers ("informational frictions"), and (3) in equilibrium, workers of the same type choose a common strategy ("symmetry frictions"). The Bayesian game, denoted by $\mathcal{G}_I$, is described formally as follows.

Players

There are $i = 1, \ldots, I$ players whom we interpret as workers.

State of nature

Each worker is endowed with a type $g \in \{1, \ldots, G\}$. The state of nature is a vector $\mathbf{g} = (g_1, \ldots, g_I)$, where g_i is a type of worker i. The probability that state $\mathbf{g}$ occurs is $\Pr(\mathbf{g}) = f_{g_1} \ldots f_{g_I}$, where $f = (f_1, \ldots, f_G)$ is some given vector of probabilities ($f_g \geq 0$, $\sum_g f_g = 1$).

Signal functions

Each worker i observes only his own type. That is, the signal function of worker i is $s_i(g_1, \ldots, g_I) = g_i$.

Actions

The set of actions of each worker is $\{1, \ldots, M\}$. A choice of action m by a worker is interpreted as a choice to apply to firm m.

Strategy

A strategy of worker i is a $G \times M$ matrix $\mathbf{p}^i = (p^i_{gm})$, where p^i_{gm} is the probability that worker i of type g applies to firm m.

Solution

In this chapter, I consider only symmetric equilibria of the game[5]

$$\widehat{p}^i_{gm} = \widehat{p}_{gm} \tag{4.3.1}$$

That is, I assume that in equilibrium each worker uses a common application strategy $\widehat{p}_{gm}$.

Payoffs

Suppose that the state of nature is $\mathbf{g} = (g_1, \ldots, g_I)$ and that worker i uses the pure strategy in which worker i applies to firm m_i with

[5] As I show in Theorem 4.3.1, the set of symmetric equilibria of $\mathcal{G}_I$ can be associated with the set of solutions of $\mathcal{M}^I$ (a solution of $\mathcal{M}^I$ is defined as a solution of the planner's constrained optimization problem 4.2.5).

probability one. Let $\mathbf{m} = (m_1, ..., m_I)$, $\mathbf{g}_{-i} = (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_I)$ and $\mathbf{m}_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_I)$. For each $m = 1, ..., M$, let $\widetilde{W}_m(\mathbf{g}, \mathbf{m})$ denote the surplus generated at firm m in the presence of worker i and, at the risk of abusing notation, let $\widetilde{W}_m^{-i}(\mathbf{g}_{-i}, \mathbf{m}_{-i})$ denote the surplus generated at firm m in the absence of worker i (the surplus is the sum of surpluses generated in all the matches in the firm where a match is formally defined in section 4.2.1 on page 103). The payoff of worker i who applies to firm m_i is

$$\widetilde{v}^i(\mathbf{g}, \mathbf{m}) = \widetilde{W}_{m_i}(\mathbf{g}, \mathbf{m}) - \widetilde{W}_{m_i}^{-i}(\mathbf{g}_{-i}, \mathbf{m}_{-i}) \tag{4.3.2}$$

Thus, worker i receives the portion of the surplus contributed by worker i.

The previous literature (for example, [27] and [34]) analyzes the Bayesian game in which (i) a single job position is traded at each market and, therefore, each Vickrey mechanism is a second price auction and (ii) the type of a worker is described by a one-dimensional characteristic. The literature describes explicetly the symmetric equilibrium and shows that the symmetric equilibrium of the Bayesian game coincides with the solution of the expected aggregate surplus maximization problem. In Theorem 4.3.1 I generalize this result for the case of $\mathcal{G}_I$ Bayesian game.

Theorem 4.3.1 *The set of solutions of the first-order Kuhn-Tucker conditions of the optimization problem 4.2.5 coincides with the set of symmetric Bayesian equilibria of* $\mathcal{G}_I$.

The intuition behind this result is straightforward. The key observation is that if the Vickrey mechanism is applied at each firm to allocate the surplus generated at the firm, then the expected payoff of each worker equals the expected aggregate surplus minus some function that does not depend on the worker's application strategy. Therefore, for each worker and for any given application strategies of the other workers, the worker's application strategy maximizes his expected payoff if and only if it maximizes the expected aggregate surplus. In the Appendix, I provide a rigorous proof of the result.

Since the inequality constraints of the optimization problem 4.2.5 are linear, first-order Kuhn-Tucker conditions must be satisfied at any solution to the optimization problem 4.2.5. This allows me to state Corollary 4.3.1.

Corollary 4.3.1 *A solution* $\widehat{\mathbf{p}}$ *to the optimization problem 4.2.5 is a symmetric Bayesian equilibrium of* $\mathcal{G}_I$*.*

When W is concave or strictly concave Theorem 4.3.1 allows us to talk about efficiency and uniqueness of the symmetric Bayesian equilibrium of $\mathcal{G}_I$ as stated in the next Corollary.

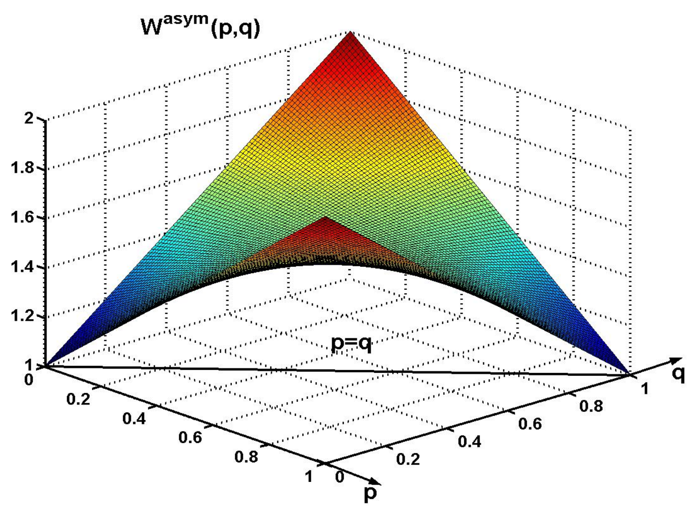
Corollary 4.3.2 *If the expected aggregate surplus function* $W(\mathbf{p})$ *is concave then any symmetric equilibrium of* $\mathcal{G}_I$ *maximizes* $W(\mathbf{p})$*. If the function* $W(\mathbf{p})$ *is strictly concave then there exists a unique symmetric equilibrium of* $\mathcal{G}_I$*.*

This generalizes the result in the previous literature that the equilibrium in the market is constrained efficient when W is concave (that is, it coincides with the solution of the constrained planner's optimization problem 4.2.5).

If, instead of defining the surplus to be a function of symmetric application probabilities, I define the surplus to be a function of asymmetric application probabilities, then the unconstrained surplus-maximizing profile of application probabilities may not be symmetric. (The asymmetric equilibrium of the Bayesian game models the market in which there are no "symmetry frictions"). In the following simple example, I consider a Bayesian game $\mathcal{G}_I$, introduced in this section, in which (i) there are two workers ($I = 2$) who are each endowed with a single common type ($G = 1$), (ii) there are two identical firms that each have a single job position at zero reservation price, and (iii) the productivity of a worker at a firm is $\omega = 1$. (There are no "informational frictions" in this game since there is only one type of worker). I compare symmetric and asymmetric equilibria of the game. Let $\mathbf{p} = (p, 1-p)$ and $\mathbf{q} = (q, 1-q)$ denote application strategies of workers one and two and let $W^{\text{asym}}(p, q)$ denote the expected aggregate surplus generated by a pair of strategies $\mathbf{p}$ and $\mathbf{q}$. The explicit expression for the expected aggregate surplus is

$$W^{\text{asym}}(p, q) =$$
$$pq + (1-p)(1-q) + 2p(1-q) + 2q(1-p) = 1 + p + q - 2pq$$

The function $W^{\text{asym}}(p, q)$ is illustrated in the figure below.



Under the symmetry constraint ($p = q$) the function $W^{\text{asym}}(p, q)$ reduces to the expected surplus function $W(p)$ as in 4.2.4. That is,

$$W(p) = 1 + 2p - 2p^2$$

The pair $(p, q) = (\frac{1}{2}, \frac{1}{2})$ is a saddle point of the function $W^{\text{asym}}(p, q)$ and $p = \frac{1}{2}$ maximizes the function $W(p)$. Function $W^{\text{asym}}(p, q)$ attains its maximum at two points: $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$. Since $W(p)$ is strictly concave, Theorem 4.3.1 implies that $(p, q) = (\frac{1}{2}, \frac{1}{2})$ is the only symmetric equilibrium of the game $\mathcal{G}_I$. Since $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$ solve

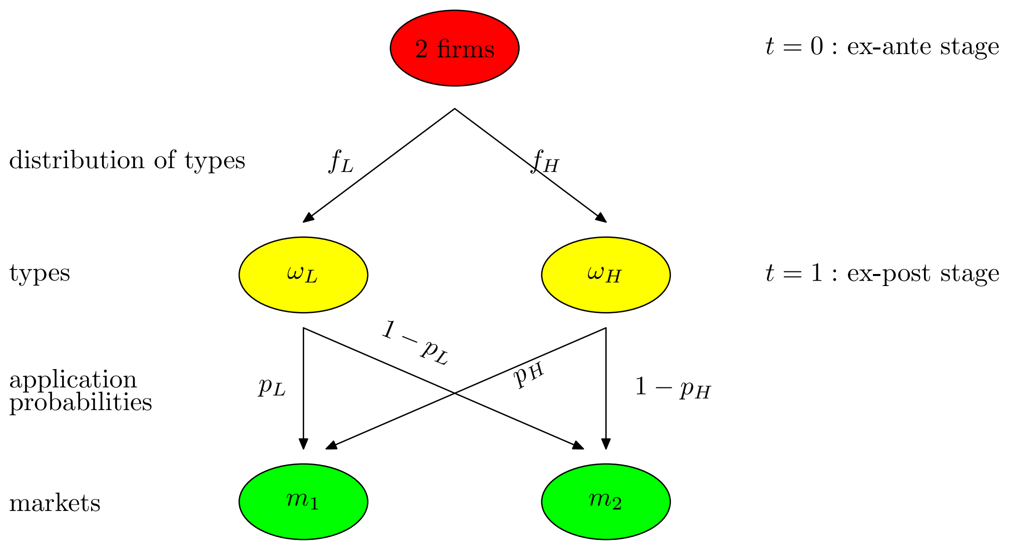
$$\begin{cases} \max_{p,q} W^{\text{asym}}(p, q) \\ p, q \in [0, 1] \end{cases} \tag{4.3.3}$$

the two strategies belong to the set of asymmetric equilibria of $\mathcal{G}_I$ (In the example, the set of Bayesian equilibria of $\mathcal{G}_I$ contains three elements: $(\frac{1}{2}, \frac{1}{2})$, $(1, 0)$, and $(0, 1)$). The maximum value of $W^{\text{asym}}(p, q)$ (attained at $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$) is larger than the maximum value of $W(p)$ (attained at $p = \frac{1}{2}$).

4.4 An Example

In Section 4.2.1 I interpret p as a $G \times M$ matrix of application probabilities (p_{gm}) that a worker of type g applies to firm m. If the number of workers who apply to a firm exceeds the number of jobs, then the many-to-one match is the one (if any) that generates the greatest positive surplus. In order to illustrate that $\mathcal{M}^I$ has applications outside of labor markets, I now present an example in which there are two firms and two markets, each of which contains 1 consumer. Each firm produces 1 indivisible unit of a good that depends on the firm's type. Each consumer demands 1 unit whose value depends on the type of firm that produced the good. In this context, p is interpreted as a $G \times M$ matrix of entrance probabilities (p_{gm}) that a firm of type g enters a market m. If two firms enter a market, then the match that occurs is the one that generates the greater surplus.

The model is described as follows. There are two firms which are identical at period $t = 0$. At period $t = 1$, the types of the firms are realized. With probability f_L, a firm's type is low, ω_L, and with probability f_H, the firm's type is high, ω_H. If the firm's type is ω_L then p_L denotes the probability that the firm enters market m_1 and $1 - p_L$ denotes the probability that it enters market m_2. If the firm's type is ω_H then p_H denotes the probability that the firm enters market m_1 and $1 - p_H$ denotes the probability that the firm enters market m_2. The aggregate surplus, generated by a matching function, equals the sum of surpluses in each market. I assume that the valuation of a consumer for the good generated by the low type firm is independent of the market but that the valuation of a consumer for the good generated by the high type firm may depend on the market. That is, a match between firm ω_L and either market generates the surplus ω_L; a match between firm ω_H and market m_1 generates the surplus ω_H and that between firm ω_H and market m_2 generates $\gamma\omega_H$ (so that market 1 is identical to market 2 when $\gamma = 1$). Therefore, the surplus, generated in market m_1, is either zero (if it is not matched with either firm), ω_L (if it is matched with a low type of firm), or ω_H (if it is matched with one high-type firm). Analogously, the surplus, generated in market m_2, is either zero, ω_L, or $\gamma\omega_H$. The model is illustrated by the following picture.



4.4.1 Optimization Problem

In this example, the expected surplus function, $W(\mathbf{p})$, can be written as follows.

$$W(\mathbf{p}) = \overbrace{f_H^2\left[p_H^2(\omega_H) + 2p_H(1-p_H)(1+\gamma)\omega_H + (1-p_H)^2(\gamma\omega_H)\right]}^{\text{both workers types are } \omega_H} +$$
$$\overbrace{f_L^2\left[p_L^2(\omega_L) + 2p_L(1-p_L)(2\omega_L) + (1-p_L)^2(\omega_L)\right]}^{\text{both workers types are } \omega_L} +$$
$$\overbrace{2f_Hf_L\left[p_Lp_H(\omega_H) + p_H(1-p_L)(\omega_H+\omega_L) + (1-p_H)p_L(\gamma\omega_H+\omega_L) + (1-p_H)(1-p_L)(\gamma\omega_H)\right]}^{\text{one worker type is } \omega_L \text{ and the other worker type is } \omega_H}$$

or equivalently, using matrix notation,

$$W(\mathbf{p}) = 2 * \begin{pmatrix} p_H, & p_L \end{pmatrix} \begin{pmatrix} -\frac{1+\gamma}{2}\omega_H f_H^2 & -f_Lf_H\omega_L \\ -f_Lf_H\omega_L & -\omega_L f_L^2 \end{pmatrix} \begin{pmatrix} p_H \\ p_L \end{pmatrix} +$$
$$+ 2\begin{pmatrix} \omega_H f_h^2 + f_Lf_H(\omega_L - (\gamma-1)\omega_H), & \omega_L f_L^2 + f_Lf_H\omega_L \end{pmatrix} \begin{pmatrix} p_H \\ p_L \end{pmatrix} +$$
$$+ \left[\omega_L f_L^2 + \gamma\omega_H f_H^2 + 2\gamma\omega_H f_L f_H\right] \quad (4.4.1)$$

If two firms enter a market, then the match that occurs is the one that generates the greater surplus. The expected surplus function satisfies the following two properties.

1. The surplus function W is concave in the matrix of entrance probabilities p [6].

2. If $\gamma = 1$, then market 1 is identical to market 2 and, in the solution, the probability that a firm enters a market is $\frac{1}{2}$. (In general, identical markets are chosen with equal probabilities by the firms of the same type in a symmetric equilibrium (see proposition 4.5.1 which is given below).)

Now, I find the entrance probabilities $(p_L, 1-p_L)$ and $(p_H, 1-p_H)$ that maximize the value of $W(\mathbf{p})$. First, I find the solution of the unconstrained optimization problem $\max_{\mathbf{p}} W(\mathbf{p})$. Then, I find the conditions under which the solution of the unconstrained optimization problem satisfies the constraints $0 \leq p_L \leq 1$ and $0 \leq p_H \leq 1$. In the case that the constraints do not hold, I construct the solution in which one of thhe constraints is binding.

The first order conditions of the optimization problem are

$$\begin{cases} f_H^2\omega_H + f_L f_H(\omega_L - (\gamma-1)\omega_H) = (1+\gamma)f_H^2\omega_H p_H + 2f_L f_H \omega_L p_L \\ f_L^2\omega_L + f_L f_H \omega_L = 2f_L^2\omega_L p_L + 2f_L f_H \omega_L p_H \end{cases}$$

The solution of the system is

$$\begin{cases} p_H = \frac{1}{2} - (\gamma-1)\frac{\frac{1}{2}+\frac{f_L}{f_H}}{1+\gamma-2\frac{\omega_L}{\omega_H}} \\ p_L = \frac{1}{2} + (\gamma-1)\frac{1+\frac{1}{2}\frac{f_H}{f_L}}{1+\gamma-2\frac{\omega_L}{\omega_H}} \end{cases} \tag{4.4.2}$$

[6] This follows since the first condition in Theorem 4.2.1 is satisfied whenever there is a single job position. Also this property can be obtained directly since the second derivative matrix (the matrix in the quadratic term of 4.4.1) has negative elements on the main diagonal and positive determinant.

The probabilities, given in 4.4.2, satisfy the constraints $0 \leq p_H \leq 1$ and $0 \leq p_L \leq 1$ whenever

$$\begin{cases} 1 \geq \frac{\omega_L}{\omega_H} + (\gamma - 1)\frac{f_L}{f_H} \\ \frac{3}{2} - \frac{\gamma}{2} \leq \frac{\omega_L}{\omega_H} + \frac{1}{2}(\gamma - 1)\frac{f_H}{f_L} \end{cases} \tag{4.4.3}$$

If the first constraint in 4.4.3 is violated, then $p_H = 0$ at the surplus maximizing entrance probabilities. If the second constraint in 4.4.3 is violated, then $p_L = 1$ at the surplus maximizing entrance probabilities.

4.4.2 Matching as a Bayesian Game

In this section, I represent $\mathcal{M}^I$ as a Bayesian game and show that the symmetric equilibrium of the game coincides with the solution of the optimization problem of the previous section. The game is described as follows.

There are two players in the game: firm one and firm two. The state of nature is described by a vector (ω_1, ω_2), where $\omega_1 \in \{\omega_L, \omega_H\}$ is the type of firm one and $\omega_2 \in \{\omega_L, \omega_H\}$ is the type of firm two. The type of the firm is interpreted as the quality of the good sold by the firm (ω_L is interpreted as a low-quality good and ω_H is interpreted as a high-quality good). Each firm observes only her own type. That is, the signal function for firm one is $s_1(\omega_1, \omega_2) = \omega_1$ and the signal function for firm two is $s_2(\omega_1, \omega_2) = \omega_2$. The set of actions for firm one and firm two is $\{m_1, m_2\}$. A firm's choice of m_j, $j = 1, 2$ is interpreted as a choice to enter market m_j. A good produced by a low-quality firm is interpreted to be a low-quality good; that produced by a high-quality firm, a high-quality good.

There is a single consumer in each market who has a perfectly elastic demand for $q = 1$ unit of the good. The low-quality good is valued at ω_L in both markets. The high-quality good is valued at ω_H in market m_1 and $\gamma\omega_H$

in market m_2, where $\gamma \geq 1$. If there is only one firm that enters a market, then the firm obtains the entire value of the good produced. If two firms of identical types enter a market, then each firm earns zero. If two firms of different types enter a market, then the low quality firm earns zero and the high quality firm earns the difference in the values.

The game can be equivalently described by the following diagram (see chapter 9 in M. Osborn "An Introduction to game Theory", 2004 for the definition and representation of a Bayesian game).

$\mathbf{1}:\omega_{\mathbf{L}}$, $\mathbf{2}:\omega_{\mathbf{L}}$ (f_L, f_L):

	m_1	m_2
m_1	$(0,0)$	(ω_L,ω_L)
m_2	(ω_L,ω_L)	$(0,0)$

$\mathbf{1}:\omega_{\mathbf{L}}$, $\mathbf{2}:\omega_{\mathbf{H}}$ (f_L, f_H):

	m_1	m_2
m_1	$(0,\omega_H-\omega_L)$	$(\omega_L,\gamma\omega_H)$
m_2	(ω_L,ω_H)	$(0,\gamma\omega_H-\omega_L)$

$\mathbf{1}:\omega_{\mathbf{H}}$, $\mathbf{2}:\omega_{\mathbf{L}}$ (f_H, f_L):

	m_1	m_2
m_1	$(\omega_H-\omega_L,0)$	(ω_H,ω_L)
m_2	$(\gamma\omega_H,\omega_L)$	$(\gamma\omega_H-\omega_L,0)$

$\mathbf{1}:\omega_{\mathbf{H}}$, $\mathbf{2}:\omega_{\mathbf{H}}$ (f_H, f_H):

	m_1	m_2
m_1	$(0,0)$	$(\omega_H,\gamma\omega_L)$
m_2	$(\gamma\omega_H,\omega_H)$	$(0,0)$

Now I show that the symmetric equilibrium of the game coincides with the solution of the optimization problem. Let $\mathbf{p}_2 = ((p_L, 1-p_L), (p_H, 1-p_H))$ be a strategy of firm two. Such a choice of $\mathbf{p}_2$ by firm two is interpreted as a decision to choose actions m_1 and m_2 with probabilities p_L and $1-p_L$ if its type is ω_L; actions m_1 and m_2, with probabilities p_H and $1-p_H$ if its type is ω_H. Given firm two's strategy $\mathbf{p}_2$ and a pure action of firm one of type ω_L, the utility of firm one is

$$u_1(m_1 \mid \omega_L) = f_L \omega_L (1 - p_L) + f_H \omega_L (1 - p_H) \tag{4.4.4}$$

if firm one chooses market m_1 and

$$u_1(m_2 \mid \omega_L) = f_L \omega_L p_L + f_H \omega_H p_H \tag{4.4.5}$$

if firm one chooses market m_2. Firm one of type ω_L is indifferent between the two actions if

$$f_L(1 - 2p_L) + f_H(1 - 2p_H) = 0 \tag{4.4.6}$$

Given firm two's strategy $\mathbf{p}_2$ and a pure action of firm one of type ω_H, the utility of firm one is

$$u_1(m_1 \mid \omega_H) = f_L \left[(\omega_H - \omega_L)p_L + \omega_H(1 - p_L)\right] + f_H \omega_H (1 - p_H) \tag{4.4.7}$$

if firm one chooses market m_1 and

$$u_1(m_2 \mid \omega_L) = f_L \left[(\gamma\omega_H - \omega_L)(1 - p_L) + \gamma\omega_H p_L\right] + f_H \gamma \omega_H p_H \tag{4.4.8}$$

if firm one chooses market m_2. Firm one of type ω_H is indifferent between the two actions if

$$f_L \left[(1 - 2p_L)\omega_L + (1 - \gamma)\omega_H\right] + f_H \left[(1 - 2p_H)\omega_H + (1 - \gamma)p_H\omega_H\right] = 0 \tag{4.4.9}$$

Solving the system of two equations 4.4.6 and 4.4.9 I obtain 4.4.2.

4.5 A Sequence of $\mathcal{M}_n^I$ Models

In section 4.6, I introduce and analyze model $\mathcal{M}^\infty$ with a continuum number of workers and firms. It is natural to formulate $\mathcal{M}^\infty$ as a limit of a sequence of models $\mathcal{M}_n^I$ with a finite number of agents. In this section, I describe element n of the sequence of models $\{\mathcal{M}_n^I, \quad n = 1, 2, \ldots, \infty\}$ in which there is (1) a finite fixed number G of worker types, (2) a finite fixed number M of firm types, (3) a number $I = n\alpha$ indicating the total number of workers (4) a fixed distribution of worker types that randomly determines the type of each worker, (5) a number $n\beta_m$ of firms of each type. In section 4.6.3, I show that $\mathcal{M}^\infty$ is naturally interpreted as a limit of the sequence $\{\mathcal{M}_n^I, \quad n = 1, 2, \ldots, \infty\}$ as n goes to infinity.

Note that the following parameters of $\mathcal{M}_n^I$ depend on n (the other parameters are constant and the same as described in the set-up of model $\mathcal{M}^I$ in section 4.2.1).

Number of agents

The number of workers is $I = n\alpha$, where α is some constant. At the risk of abusing notation, I now use index m to denote the type of a firm and I use index M to denote the number of firm types. The number of firms of type m is $n\beta_m$ and the total number of firms is $n \sum_m \beta_m$. All firms of type m are identical. That is, the number of job positions, the reserve values of the job positions, and the productivity of workers are the same at any two firms of type m. I assume also that for each m

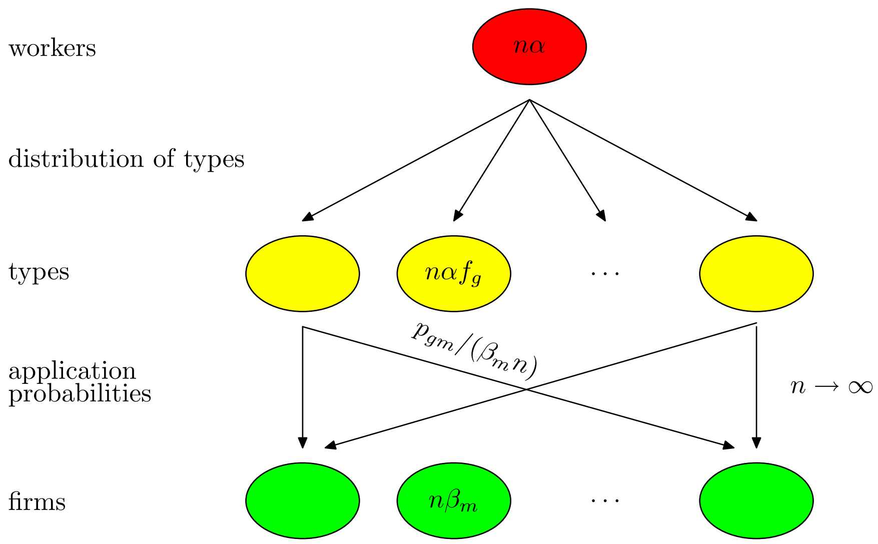
$$r_{jm} = r_m \quad \text{for any } j \tag{4.5.1}$$

so that, by Theorem 4.2.1, the surplus function $W(\mathbf{p})$ associated with $\mathcal{M}_n^I$ is concave.

Application probabilities

The general matrix of application probabilities associated with the model $\mathcal{M}_n^I$ is of size $G \times n \sum_m \beta_m$. However, Proposition 4.5.1 shows that I can restrict the matrix of application probabilities to be symmetric with respect to the probability that a worker applies to any firm of type m (note that the proposition requires concavity of the surplus function $W(\mathbf{p})$. That is, while looking for the surplus maximizing matrix of application probabilities, Proposition 4.5.1 allows me to restrict to those matrices in which a worker of type g applies with equal probability to any firm of type m. This simplifies the analysis and allow me to consider matrices of smaller dimension. Thus, I need only consider $G \times M$ matrices $\mathbf{p} = (p_{gm})$ such that $p_{gm} \geq 0$ and $\sum_m p_{gm} = 1$ where p_{gm} is the probability that a worker of type g applies to a firm of type m and the probability that a worker of type g applies to a specific firm of type m is $\frac{p_{gm}}{n\beta_m}$.

For each n, the parameters $\left(G, M, \alpha, \beta_m, \omega_{gm}, r_m, K_m, f, p_{gm}\right)$ of $\mathcal{M}_n^I$ are independent of n. The model$\mathcal{M}_n^I$ is illustrated in the following figure.



Proposition 4.5.1 *Let S be a set of firms such that for any $m \in S$ there exist constants K_S ,r_S, ω_{gS}, such that $K_m = K_S$, $r_m = r_S$, $\omega_{gm} = \omega_{gS}$. Then there exists a surplus-maximizing matrix of application probabilities such that*

for each g workers of type g apply with equal probabilities to the firms in this set, that is, $p_{gm} = p_{gS}$ *for each firm of type m that belongs to the set S.*

Thus, whenever there is a set of firms that are identical from the point of view of all workers ex ante, there exists a surplus-maximizing probability matrix in which a worker's surplus maximizing probability of applying to each firm in the set is constant across all firms in the set. I give some intuition in the case that the set of identical firms is the entire set of firms. Recall that the set of solutions to the optimization problem 4.2.5 is identical to the set of Bayesian equilibria of the Bayesian game $\mathcal{G}_I$. If the firms are identical from the point of view of all workers ex ante, and all workers apply to each firm with a common probability then, all workers are indifferent among all firms so there exists an equilibrium of the Bayesian game $\mathcal{G}_I$ in which the probability that a worker applies to a firm is constant across firms. The proof is in the appendix. Corollary 4.5.1 shows that the description of worker types should depend only on productivity characteristics.

Corollary 4.5.1 *Let S be a set of types of workers such that there exist constants,* ω_{Sm}*, such that* $\omega_{gm} = \omega_{Sm}$ *for any worker whose types g belongs to the set S. Then there exists a surplus-maximizing matrix of application probabilities such that for each m the application probability* $p_{gm} = p_{Sm}$ *so that each worker whose type g belongs to the set S applies to firm m with a common probability.*

Thus, whenever there is a set of types of workers for whom the productivity of the worker depends only on the type of firm and not on the worker's type, there exists a surplus-maximizing probability matrix for which the surplus maximizing probability of applying to firm m is constant across all types of workers in the set. Intuition analogous to that given above can be offered.

In the literature, it is common for frictions to be introduced in models with a finite number of types. Generally, if there is a continuum of types, every agent is different and so different types choose different firms which implies that there are no frictions. However, in this paper, it is possible to formulate a model with frictions in which there is a continuum number of types and agents as a limit of models with frictions in which there is a finite number of types and agents[7]. Therefore, coordination frictions can be naturally defined in a model with a continuum number of types. In the next section, I show how a model with frictions in which there are a continuum of workers and firms and a finite number of types can be viewed as a limit of the sequence $\mathcal{M}_n^I$.

4.6 Continuous Model

In this section, I consider a model, denoted by $\mathcal{M}^\infty$, in which there is a continuum of workers of each type and a continuum of firms of each type. The number of types of workers and firms is finite. The mass of workers of type g is αf_g which is denoted by α_g. The mass of firms of type m is β_m. The masses can be interpreted naturally as the limit of sequences of normalized masses of the model$\mathcal{M}_n^I$ as $n \to \infty$. To see this, normalize the mass of workers and firms in $\mathcal{M}_n^I$ of a given type by dividing the number of workers and firms of each type by n. The number of firms of type m in $\mathcal{M}_n^I$ is deterministic and equal to $n\beta_m$. Therefore, the mass of firms of type m is β_m and the limit is β_m as $n \to \infty$. The number R_g of workers of type g in $\mathcal{M}_n^I$ is a random variable with expectation $n\alpha_g$. By the law of large numbers, as $n \to \infty$, the normalized random variable R_g converges to its expectation,

[7] I do not consider the model with a continuum number of types in this paper and the results of this paper do not apply directly to this case.

α_g. Therefore, one can interpret the masses of workers and firms in $\mathcal{M}^\infty$ as a limit of normalized masses in $\mathcal{M}_n^I$.

4.6.1 Model Set Up

There is no ex-ante stage and, therefore, there is no uncertainty about the workers' types.

Agents

Worker types are denoted by $g \in \{1, \ldots, G\}$ so that there are G types of workers. The mass of workers of type g is α_g. Firm types are denoted by $m \in \{1 \ldots M\}$ so that there are M types of firms. The mass of firms of type m is β_m. Let K_m denote the number of job positions available at each firm of type m and let r_{jm} denote the reserve value of job $j = 1, \ldots, K_m$ at each firm of type m. As in $\mathcal{M}_n^I$, I assume that for each m

$$r_{jm} = r_m \quad \text{for any } j \tag{4.6.1}$$

Preferences

As in $\mathcal{M}^I$, I assume that agents are risk-neutral and that utilities of the agents are transferrable. Workers have zero reservation utilities.

Flow parameters

I cannot define directly the probability that a worker of type g applies to a specific firm[8] of type m in $\mathcal{M}^\infty$. Instead, I introduce the following notion of flows of workers to jobs. Recall that in $\mathcal{M}_n^I$, we can restrict the matrix of application probabilities to be symmetric with respect to the probability that a worker applies to any firm of type m. The

[8] In $\mathcal{M}_n^I$, the probability that a worker of type g applies to a specific firm of type m is defined as $\frac{p_{gm}}{n\beta_m}$. In the limit the probability is equal to zero.

number of workers is a random variable whose expectation is $n\alpha_g$. Each worker of type g applies with probability $\frac{p_{gm}}{n\beta_m}$ to a firm of type m. It is well known[9] that as $n \to \infty$ the distribution of the number of workers of type g who arrive at a firm of type m converges to the Poisson distribution with parameter

$$q_{gm} = \frac{\alpha_g}{\beta_m} p_{gm} \tag{4.6.2}$$

Therefore, I assume that, in $\mathcal{M}^{\infty}$, the probability that exactly $x \in \{0, 1, \ldots\}$ workers of type g apply to a firm of type m is

$$\nu_x = \frac{(q_{gm})^x}{x!} \exp(-q_{gm}) \tag{4.6.3}$$

where the flow parameters q_{gm} (parameter q_{gm} is the average number of applications from the workers of type g to a specific firm of type m) satisfy the constraints[10]

$$\begin{cases} \sum_m \beta_m q_{gm} = \alpha_g & \text{for each } g \\ q_{gm} \geq 0 & \text{for all } g, m \end{cases} \tag{4.6.4}$$

The constraint 4.6.4 is a *mass balance* condition. It says that the mass of workers of type g equals the mass of applications to all the firms from workers of type g. (This is an analogue of the condition that each worker applies to a single firm in the discrete case).

Surplus

Pick a firm of type m. Suppose that the number of workers of type g who apply to this firm of type m is denoted by x_{gm}. Given $\mathbf{x}_m = (x_{1m}, \ldots, x_{Gm})$, the surplus function generated at the firm of type

[9] The result can be found in most probability theory textbooks (see for example, V. Rotar, "Probability Theory", 1997 (chapter 10)).

[10] The constraints are induced by the constraints on application probabilities $p_{gm} \geq 0$ and $\sum_m p_{gm} = 1$ in $\mathcal{M}_n^I$.

m, denoted by $\widetilde{W}^c_m(\mathbf{x}_m)$, is the sum of surpluses generated in all the matches in the firm[11]. Given $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_M)$, the aggregate surplus, denoted by $\widetilde{W}^c(\mathbf{x})$, is the sum of surpluses generated in all the firms

$$\widetilde{W}^c(\mathbf{x}) = \sum_m \beta_m \widetilde{W}^c_m(\mathbf{x}_m) \tag{4.6.5}$$

Given specific flow parameters we can take expectations to calculate the expected surplus generated at each firm of type m and then add up to obtain the expected aggregate surplus generated by all firms of each type $m = 1, .., M$. Let $q_m = (q_{1m}, \ldots, q_{Gm})$ denote the vector of flow parameters of type $g = 1, \ldots, G$ workers to a firm of type m. Let $\mathbf{q} = (q_{gm})$ denote the $G \times M$ matrix of flow parameters. The expected surplus, denoted by $W^c_m(\mathbf{q}_m)$, generated at a firm[12] of type m is the expectation of $\widetilde{W}^c_m(\mathbf{x}_m)$ with respect to the realizations of $\mathbf{x}_m$ given $\mathbf{q} = (q_{gm})$.

$$W^c_m(\mathbf{q}) = \mathtt{E}_{\mathbf{x}_m}\left(\widetilde{W}^c_m(\mathbf{x}_m) \mid \mathbf{q}\right) \tag{4.6.6}$$

An explicit expression of the function $W^c_m(\mathbf{q}_m)$ is given in section 4.8.1 in the appendix of this paper. Finally, the expected aggregate surplus, denoted by $W^c(\mathbf{q})$, is the expectation of $\widetilde{W}^c(\mathbf{x})$ with respect to the realizations of $\mathbf{x}$ given $\mathbf{q} = (q_{gm})$.

$$W^c(\mathbf{q}) = \mathtt{E}_{\mathbf{x}}\left(\widetilde{W}^c(\mathbf{x}) \mid \mathbf{q}\right) \tag{4.6.7}$$

Solution Concept

[11] As in $\mathcal{M}^I$, the surplus generated in a match between a worker of type g and job j in a firm of type m is $\omega_{gm} - r_m$. The matches in a firm are described in section 4.2.1 on page 103

[12] The surplus depends only on the parameters of the flows of workers to firms of type m and does not depend on the parameters of the flows of workers to firms of other types.

The solution of the model is an array $\mathbf{q} = (q_{gm})$ of flow parameters such that $\mathbf{q}$ maximizes the expected aggregate surplus $W^c(\mathbf{q})$. The formal description of the optimization problem is given in the next section.

4.6.2 Optimization Problem

The surplus maximizing $G \times M$ matrix $\mathbf{q} = (q_{gm})$ of workers' flows parameters is found from the following maximization problem

$$\begin{cases} \max_{\mathbf{q}} W^c(\mathbf{q}) \\ \sum_m \beta_m q_{gm} = \alpha_g \\ q_{gm} \geq 0 \end{cases} \tag{4.6.8}$$

where $W^c(\mathbf{q}) = \sum_m \beta_m W^c_m(\mathbf{q}_m)$.

So as to have an idea of what the objective function might look like, I now give an explicit expression for the expected surplus $W^c_m(\mathbf{q}_m)$ in the case that the number of jobs at a firm of type m is $K_m = 1$ for each m. Let π_m denote a permutation

$$\pi_m = \begin{pmatrix} 1 \, , \ldots , \; G \\ \pi_{1m}, \ldots , \pi_{Gm} \end{pmatrix} \tag{4.6.9}$$

such that the productivity of workers of different types at a firm of type m is ordered as follows

$$\omega_{\pi_{1m}m} \geq \omega_{\pi_{2m}m} \geq \ldots \geq \omega_{\pi_{Lm}m} \geq r_m \geq \ldots \geq \omega_{Gm} \geq 0 \tag{4.6.10}$$

Then with probability $(1 - \exp(q_{\pi_{1m}m}))$ at least one worker with productivity $\omega_{\pi_{1m}m}$ applies to the firm and generates surplus $\omega_{\pi_{1m}m}$. With probability $\exp(-q_{\pi_{1m}m})(1 - \exp(q_{\pi_{2m}m}))$ no worker of productivity $\omega_{\pi_{1m}m}$ and at least one worker of productivity $\omega_{\pi_{2m}m}$ applies to the firm and the match generates surplus $\omega_{\pi_{2m}m}$. In general, whenever $g \leq L$ the probability that surplus $\omega_{\pi_{gm}m}$ is generated in the firm is $\exp(-q_{\pi_{1m}m} - \ldots - q_{\pi_{g-1m}m})(1 - \exp(q_{\pi_{gm}m}))$.

Whenever $g > L$ a worker of type $\omega_{\pi_{gm}m}$ never obtains the job position at a firm of type m. Taking expectation, I obtain

$$W_m^c(\mathbf{q}_m) = \omega_{\pi_{1m}m}(1 - \exp(q_{\pi_{1m}m})) + \omega_{\pi_{2m}m}\exp(-q_{\pi_{1m}m})(1 - \exp(q_{\pi_{2m}m}))$$
$$+ \ldots + \omega_{\pi_{Lm}m}\exp(-q_{\pi_{1m}m} - \ldots - q_{\pi_{L-1m}m})(1 - \exp(q_{\pi_{Lm}m})) =$$
$$(\omega_{\pi_{1m}m} - \omega_{\pi_{2m}m})(1 - \exp(-q_{\pi_{1m}m})) + (\omega_{\pi_{2m}m} - \omega_{\pi_{3m}m})(1 - \exp(-q_{\pi_{1m}m} - q_{\pi_{2m}m}))$$
$$+ \ldots + \omega_{\pi_{Lm}m}(1 - \exp(-q_{\pi_{1m}m} - \ldots - q_{\pi_{Lm}m})) \qquad (4.6.11)$$

4.6.3 Continuous Model as a Limit of a Sequence of Discrete Models

In Theorem 4.6.1 I show the relationship between the optimization problem associated with $\mathcal{M}^\infty$ and that associated with$\mathcal{M}_n^I$. Suppose that the surplus generated at a firm of type m in $\mathcal{M}_n^I$ is denoted by $W_m^{(n)}(\mathbf{p}_m)$ and that in $\mathcal{M}^\infty$ is denoted by $W_m^c(\mathbf{q}_m)$.

Theorem 4.6.1 *Suppose that the matrix of application probabilities* $\mathbf{p}$ *in* $\mathcal{M}_n^I$ *and the matrix of flow parameters* $\mathbf{q}$ *in* $\mathcal{M}^\infty$ *satisfy*

$$p_{gm} = \frac{\beta_m}{\alpha_g} q_{gm} \qquad (4.6.12)$$

1. *As* $n \to \infty$, $W_m^{(n)}(\mathbf{p}_m) \to W_m^c(\mathbf{q}_m)$ *and, therefore,* $\frac{W^{(n)}(\mathbf{p})}{n} \to W^c(\mathbf{q})$ *where the convergence is pointwise.*

2. *If* $W^{(n)}(\mathbf{p})$ *is concave with respect to* $\mathbf{p}$ *for any* n *then* $W^c(\mathbf{q})$ *is concave with respect to* $\mathbf{q}$.

The proof can be found in section 4.8.4 in the appendix of the paper.

4.6.4 Construction of the Optimal Flows in the Case of a Single Job Position at each Firm ($K_m = 1$)

By Theorem 4.2.1, the expected aggregate surplus function $W^{(n)}(\mathbf{p})$ is concave with respect to $\mathbf{p}$ (if $K_m = 1$ for any m then condition (a) of the theorem holds). Therefore, by Theorem 4.6.1, the function $W^c(\mathbf{q})$ is also concave. Thus, a variety of standard methods[13] can be applied to construct the solution of problem 4.6.8. An explicit expression for $W^c_m(\mathbf{q}_m)$ is given in 4.6.11.

In this section, I apply one such procedure to a modified but equivalent optimization problem. The procedure has a natural interpretation of an ascending bid auction[14] in which firms bid for the flows of the workers.

Since the functions $W^c_m(\mathbf{q}_m)$ are concave, problem 4.6.8 can be equivalently represented as the following min max problem.

$$\min_u \left[\max_{q \geq 0} \sum_m \beta_m \left(W^c_m(\mathbf{q}_m) - \sum_g u_q q_{gm} \right) + \sum_g u_g \alpha_g \right] \tag{4.6.13}$$

where $u = (u_1, \ldots, u_g)$ is a vector of lagrange multipliers associated with the constraints $\sum_m \beta_m q_{gm} = \alpha_g$ in 4.6.8.

Let

$$\phi(u) = \max_{q \geq 0} \sum_m \beta_m \left[W^c_m(\mathbf{q}_m) - \sum_g u_q q_{gm} \right] + \sum_g u_g \alpha_g \tag{4.6.14}$$

The function $\phi(u)$ is a maximum of functions that are linear with respect to u. Therefore, the function $\phi(u)$ is concave. Thus, the constrained min max problem 4.6.13 can be rewritten as the unconstrained concave minimization problem

$$\min_u \phi(u) \tag{4.6.15}$$

[13]See, for example, Hoang Tuy, "Convex Analysis and Global Optimization"

[14]This is a natural analogue of a similar procedure applied in the case of a frictionless market which is described by an associated assignment linear optimization problem. See, for example, [30] and [6]

In the rest of this section I make the following assumptions.

Assumption 4.6.1
(i) $\omega_{gm} > r_m$ *for any* g *and* m
(ii) *for each* m, $\omega_{gm} \neq \omega_{\widetilde{g}m}$ *for any* $g \neq \widetilde{g}$

Without loss of generality, I also assume that $r_m = 0$ for all m.

Properties of Function $\phi(u)$

In this section, I derive some important properties of $\phi(u)$. I use these properties to prove that, at each iteration, the Auction algorithm, described in section 4.6.4, produces variables that converge to the surplus-maximizing matrix of flow parameters $\widehat{\mathbf{q}}$ and associated lagrange multipliers $\widehat{u}$.

First, I show that the function $\phi(u)$ is differentiable and derive explicit expressions for the first derivative of $\phi(u)$.

Lemma 4.6.1 *If Assumption 4.6.1 holds then the function* $W_m^c(\mathbf{q}_m)$ *is strictly concave and, therefore, for any vector* $u = (u_1, \ldots, u_G)$ *there exists a unique solution* $\widehat{\mathbf{q}}_m$ *to the optimization problem*

$$\max_{\mathbf{q}_m \geq 0} \left[W_m^c(\mathbf{q}_m) - \sum_g u_q q_{gm} \right] + \sum_g \alpha_g u_g \tag{4.6.16}$$

By Lemma 4.6.1, the function $\phi(u)$ is differentiable. Let $q_{gm}(u)$ denote the solution of problem 4.6.16 for a given vector u. Using the Envelope theorem applied to 4.6.16 I derive the following vector of the partial derivatives of the function $\phi(u)$.

$$\nabla_g \phi(u) = \alpha_g - \sum_m \beta_m q_{gm}(u) \tag{4.6.17}$$

The next two lemmas establish some properties of the functions $q_{gm}(u)$ and $\nabla_g \phi(u)$.

Lemma 4.6.2 *The solution $q_{gm}(u)$ to the optimization problem 4.6.16 has the following properties.*

1. *For each g, the function $q_{gm}(u_1, \ldots, u_G)$ is a non-increasing function of u_g.*

2. *For each g, the function $q_{gm}(u_1, \ldots, u_G)$ is a non-decreasing function of u_h for any $h \neq g$.*

The next lemma follows immediately from Lemma 4.6.2 and equation 4.6.17 for the first derivative of $\phi(u)$.

Lemma 4.6.3 *For each g, the partial derivative $\nabla_g \phi(u_1, \ldots, u_G)$ has the following properties.*

1. *$\nabla_g \phi(u_1, \ldots, u_G)$ is a non-decreasing function of u_g.*

2. *$\nabla_g \phi(u_1, \ldots, u_G)$ is a non-increasing function of u_h for any $h \neq g$.*

Auction Algorithm

Recall that since I have shown that the function $W^c(\mathbf{q})$ is concave there is a variety of standard methods that can be applied to construct the solution of the constrained optimization problem 4.6.8. Recall also that the constrained optimization problem 4.6.8 is equivalent to the unconstrained optimization problem 4.6.15. I have also shown that the objective function of the unconstrained optimization problem 4.6.15 satisfies certain properties. These properties allow me to apply an Auction algorithm to the unconstrained optimization problem 4.6.15. The Auction algorithm constructs numerically the solution to the the unconstrained optimization problem 4.6.15. By equivalence, the solution is also the solution to the constrained optimization problem 4.6.8. The Auction algorithm is an example of a Gauss-Seidel method[15].

[15] Some standard references on Gauss-Seidel method are [5] and [25]

The algorithm has a natural economic interpretation. The procedure describes an auction in which the firms bid for the flows of workers of different types by posting the wages they pay to different types of workers. The auction starts with the auctioneer submitting a $1 \times G$ vector of wages $(0, ..., 0)$ which are then increased gradually during the auction as follows. At zero wages, the expected positive mass of workers of type g demanded by firms is larger than the positive mass of workers of type g who desire jobs and so the auctioneer increases the wages in some way. If only the wage of workers of type g increase then demand for workers of type g decreases and the demand for workers of other types increases. The auction stops when there is no excess demand for any type of workers. Formally, the algorithm is described as follows.

Algorithm 4.6.1 (Auction Algorithm)

step 1. *Start with zero workers' utilities* $u_g = 0 \quad g = 1, \ldots, G$.

step 2. *For given utilities* $\mathbf{u} = (u_1, \ldots, u_G)$, *find solution of the optimization problem 4.6.16. The solution* $\mathbf{q}_m(\mathbf{u}) = (q_{1m}(\mathbf{u}), \ldots, q_{Gm}(\mathbf{u}))$ *is interpreted as demand of a firm of type* m *for different types of workers.*

step 3. *Find index* g *such that the following inequality holds*

$$\sum_m \beta_m q_{gm} > \alpha_g \tag{4.6.18}$$

The left-hand side of the equation is interpreted as the aggregate demand for workers of type g. *The right-hand side of the equation is interpreted as the supply of workers of type* g. *If no such inequality exists then stop. If there is at least one such inequality, increase the corresponding value* u_g *by some small* $\epsilon > 0$ *and return to step 2.*

Figure 4.6.1 illustrates convergence of the auction algorithm in the case of two types of workers, $g = 1, 2$.

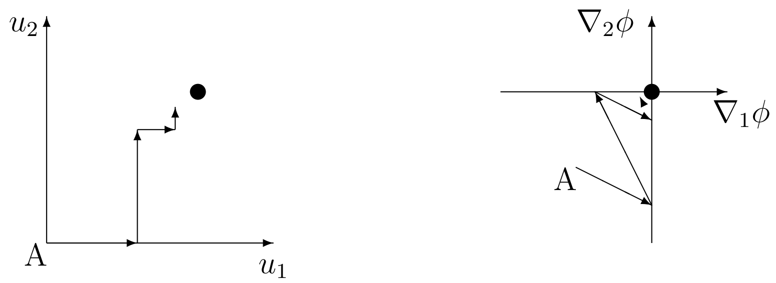


Figure 4.6.1 *The figure shows convergence of the algorithm in* (u_1, u_2) *and* $(\nabla_1\phi, \nabla_2\phi)$ *coordinates. The algorithm starts at point A where* $u_1 = u_2 = 0$ *and* $\nabla_1\phi < 0$, $\nabla_2\phi < 0$. *By Lemma 4.6.3, as I increase* u_1 *the value of* $\nabla_1\phi$ *increases and the value of* $\nabla_2\phi$ *decreases. I increase* u_1 *until* $\nabla_1\phi = 0$. *Then I increase* u_2 *until* $\nabla_2\phi = 0$. *At any stage of the procedure both* $\nabla_1\phi$ *and* $\nabla_2\phi$ *are negative or zero and* u_1 *and* u_2 *are increasing. However,* u_1 *and* u_2 *can not be increasing till infinity (if* $u_g \to \infty$ *then* $\nabla_g\phi \to \alpha_g > 0$*). Therefore, the procedure converges to the point where* $\nabla_1\phi = \nabla_2\phi = 0$.

Proposition 4.6.1 *The Auction algorithm 4.6.1 produces a* $G \times M$ *matrix* $(\widehat{q}_{gm})$ *of demands and a vector of wages* $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_G)$ *such that the* $G \times M$ *matrix* $(\widehat{q}_{gm})$ *solves the constrained maximization problem 4.6.8 and the vector of wages is the associated vector of Lagrange multipliers.*

4.7 Conclusion and Extensions

The paper studies the problem of constructing the equilibrium flow of workers to jobs in a model of the labor market with frictions, a finite number of types available for each agent, a general form of heterogeneity of the agents' types, and a finite number of homogeneous job positions at each firm. The problem of workers' allocation to jobs in the labor market with frictions is modelled as a Bayesian game. The paper shows that the the equilibrium flow can be constructed as a solution of an associated planner's constrained concave optimization problem. The paper proposes a new use of standard numerical procedures in constructing the Bayesian equilibrium of the model.

To summarize, the paper shows how standard numerical procedures can be used to maximize the surplus of the labour model and do comparative statics in the labor market with coordination frictions. The next natural step would be to find some interesting examples, that can be described by the model, and demonstrate how the techniques actually work. Also to make comparative statics meaningful we need conditions for the uniqueness of the equilibrium that are easily verifiable in practice. These are two of the main priorities in my future research.

Many extensions of the model are natural. I list here only a few of them. The model describes a static matching model. An important question is whether the static model is a steady state of some dynamic model of the labor market. This would make the model more useful for various applications.

Another assumption that I want to relax is that the job positions are homogeneous in each firm. Because of the assumption, only markets for a specific specialty can be described by the model. In practice, however, a firm hires people with different specialties. It is easy to extend the model set-up to this case but much more difficult to prove that the properties of the optimization problem hold. (At this moment I also do not know the counterexamples that show, for instance, that planner's optimization problem is not concave).

In the case of a continuum agents of each type and a single job position at each firm I have shown that stronger properties of the optimization problem hold. The properties guarantee, for example, that the auction algorithm produces the solution of the optimization problem. It would be interesting to verify whether the properties hold in a more general case.

Finally, the model can also be extended to the non transferable utility case. It can be shown that the equilibrium outcome can be obtained as a solution to a system of linear equations. To show how the equilibrium can be constructed we need to analyze the properties of the system.

4.8 Appendix

4.8.1 Representation of the Objective Function in $\mathcal{M}^I$ and $\mathcal{M}^\infty$ under Different Restrictions on the Parameters of the Models

In this section, I give an explicit description of $W_m(\mathbf{p}_m)$ and $W_m^c(\mathbf{q}_m)$ functions under different restrictions on the parameters in $\mathcal{M}^I$ and $\mathcal{M}^\infty$.

Description of $W_m(\mathbf{p}_m)$ in $\mathcal{M}^I$

Recall the definition of the function $W_m(\mathbf{p}_m)$ in $\mathcal{M}^I$. First, each worker i draws randomly his type. With probability f_g the realized type is g. Given type g, worker i applies with probability p_{gm} to firm m. As a result, some random set $\mathcal{I}_m$ of workers applies to firm m. Suppose that the workers in the set $\mathcal{I}_m$ are ordered in such a way that their productivity is decreasing and that the job positions at firm m are ordered in such a way that the reserve values of the jobs are increasing, $r_{1m} \geq r_{2m} \geq \ldots \geq r_{K_m m}$. The workers in the set $\mathcal{I}_m$ and the jobs at firm m are matched in a one-to-one fashion so that the worker with the highest productivity is matched to a job with the lowest reserve value, the worker with the second highest productivity is matched with the job with the second highest reserve value, etc. Each match between a worker of type g and a job position j at firm m

generates surplus $\max(\omega_{gm} - r_{jm}, 0)$, denoted by $(\omega_{gm} - r_{jm})^+$. To describe analytically the surplus generated at firm m, I associate the following random variable X_i with each worker i. With probability $1 - \sum_g f_g p_{gm}$ (worker i does not apply to firm m) $X_i = 0$. With probability $f_g p_{gm}$ (worker i draws type g and applies to firm m) $X_i = \omega_{gm}$. Workers are ex-ante identical, draw their type and apply to firms independently of each other, and use symmetric application probabilities. Therefore, the random variables X_i are independent and identically distributed. To determine the matches in the firm I order the sequence $X_1, \ldots, X_I$ in a decreasing order so that $X^{(1)} \geq X^{(2)} \geq \ldots \geq X^{(I)}$. Random variable $X^{(j)}$ is a j^{th} largest element in the sequence $X_1, \ldots, X_I$. By definition, the surplus generated at firm m is

$$W_m(\mathbf{p}_m) = \mathtt{E} \sum_{j=1}^{K_m} \left(X^{(j)} - r_{jm}\right)^+ \tag{4.8.1}$$

Now I impose different restrictions on the parameters of the model to simplify 4.8.1. I consider three different cases.

Case 1

Let's assume that all the job positions at firm m have the same reserve value. Formally, this condition can be written as

Assumption 4.8.1

$$\textit{for each } m, \quad r_{jm} = r_m \quad \textit{for all } j \tag{4.8.2}$$

Without loss of generality, I assume that $r_m = 0$. It follows immediately from 4.8.1 that the function $W_m(\mathbf{p}_m)$ can be represented as follows.

Proposition 4.8.1 *Under Assumption 4.8.1 the function $W_m(\mathbf{p}_m)$ can be represented as follows. Let $X_1, X_2, \ldots, X_I$ be independent identically distributed random variables. Suppose that for each i support of X_i is $\{0, \omega_{1m}, \ldots, \omega_{Gm}\}$ and distribution $h = (h_0, h_1, \ldots, h_G)$ of X_i is*

$$h_g = Pr(X_i = \omega_{gm}) = f_g p_{gm} \qquad h_0 = Pr(X_i = 0) = 1 - \sum_g f_g p_{gm} \quad (4.8.3)$$

Then

$$W_m(\mathbf{p}_m) = \mathtt{E} \sum_{j=1}^{K_m} X^{(j)} \quad (4.8.4)$$

where $X^{(j)}$ is the j^{th} largest value in the sequence $X_1, X_2, \ldots, X_I$.

<u>**Case 2**</u>

Suppose that the productivity of a worker at firm m does not depend on the type of the worker. Formally, this condition can be written as follows.

Assumption 4.8.2

$$\text{for each } m, \quad \omega_{gm} = \omega_m \quad \text{for any } g \quad (4.8.5)$$

In this case each random variable X_i can take only two values, either zero or ω_m. Let's call a realization of ω_m a success and define random variable Y as the number of successes in $X_1, \ldots, X_n$ sequence (Formally, $Y = \frac{1}{\omega_m} \sum_i X_i$). Without loss of generality, I assume that $K_m = \infty$ (The case of a finite K_m is equivalent to the case in which $K_m = \infty$ and $r_{jm} = \infty$ for $j > K_m$). Then the surplus function $W_m(\mathbf{p}_m)$ in 4.8.1 can be represented as

$$W_m(\mathbf{p}_m) = \mathtt{E} \sum_{j=1}^{Y} (\omega_m - r_{jm})^+ \quad (4.8.6)$$

where random variable Y has binomial distribution with parameters

$$Y \sim B(I, \sum_g f_g p_{gm}) \quad (4.8.7)$$

Alternatively, the function $W_m(\mathbf{p}_m)$ can be described as follows.

Proposition 4.8.2 *Under Assumption 4.8.2 The function $W_m(\mathbf{p}_m)$ can be described as follows. Let's consider the following experiment. There is an infinite staircase and a lady standing at the bottom of the staircase. The lady flips a coin I times. Each time the coin shows heads, the lady goes one step up. Each time the coin shows tails, she does not move. The probability of heads is $p = \sum_g f_g p_{gm}$. Then the function $W_m(\mathbf{p}_m)$ is equal to the expected height that the lady goes up.*

Note that the size of step j of the staircase corresponds to the value $(\omega_m - r_{jm})^+$ in 4.8.6 and the size of the step is decreasing with j.

<u>**Case 3**</u>

Let's assume that there is a single job position at each firm. Formally, this can be written as

Assumption 4.8.3

$$\textit{for each } m, \quad K_m = 1 \tag{4.8.8}$$

This restriction is stronger than the restriction in case one. The expression for $W_m(\mathbf{p}_m)$ given in 4.8.4 can be simplified as

$$W_m(\mathbf{p}_m) = \mathtt{E}X^{(1)} \tag{4.8.9}$$

where $X^{(1)} = \max(X_1, \ldots, X_I)$. If I substitute the explicit expression for $\mathtt{E}X^{(1)}$ in 4.8.9 I obtain

$$W_m(\mathbf{p}_m) = \sum_{g:\omega_{gm} \geq r_m} \left(1 - \left(1 - \sum_{g':\omega_{g'm} \geq \omega_{gm}} f_{g'm} p_{g'm}\right)^I\right) \triangle\omega_{gm} \tag{4.8.10}$$

Therefore, $W_m(\mathbf{p}_m)$ is a function of the form

$$W_m(\mathbf{p}_m) = \sum_g \theta_g \gamma(A^g \mathbf{p}_m) \tag{4.8.11}$$

where $\mathbf{p}_m$ is a G by 1 vector, A^g are some 1 by G vectors, θ_g are some positive constants, and

$$\begin{aligned} \gamma(x) &: \mathbb{R} \to \mathbb{R} \\ \gamma(x) &= 1 - (1-x)^I, \quad x \in [0,1] \end{aligned}$$

It is straightforward to verify that the function $W_m(\mathbf{p}_m)$ of the form given in 4.8.11 is a concave function of $\mathbf{p}_m$.

Description of $W_m^c(\mathbf{q}_m)$ in $\mathcal{M}^\infty$

Recall the definition of $W_m^c(\mathbf{q}_m)$ in $\mathcal{M}^\infty$. Let's pick a specific firm of type m. For each type g the number of workers of type g who apply to the firm of type m is a random variable variable, denoted by x_g. Random variables $x_1, \ldots, x_G$ are independent and distributed according to Poisson distribution with parameter q_{gm}. The productivities of the workers who apply to the firm are described by the following sequence, denoted by ω

$$\omega = \left(\overbrace{\omega_{1m}^1, \ldots, \omega_{1m}^{x_1}}^{x_1 \text{ copies of } \omega_{1m}}, \overbrace{\omega_{2m}^1, \ldots, \omega_{1m}^{x_2}}^{x_2 \text{ copies of } \omega_{2m}}, \ldots, \overbrace{\omega_{Gm}^1, \ldots, \omega_{Gm}^{x_G}}^{x_G \text{ copies of } \omega_{Gm}} \right) \tag{4.8.12}$$

Suppose that the $\sum_g x_g$ elements of the sequence ω are ordered in a decreasing order so that

$$\omega^{(1)} \geq \omega^{(2)} \geq \ldots \geq \omega^{(\sum_g x_g)} \tag{4.8.13}$$

where $\omega^{(j)}$ is a j^{th} largest element in sequence ω. The surplus, generated in the firm of type m, can be written as follows.

$$W_m^c(\mathbf{q}) = \mathtt{E} \sum_{j=1}^{\min(K_m, \sum_g x_g)} \left(\omega^{(j)} - r_{jm}\right)^+, \tag{4.8.14}$$

where, as before, r_{jm} is an increasing sequence of the reserve values of job positions at the firm of type m and $(\omega^{(j)} - r_{jm})^+ = \max(\omega^{(j)} - r_{jm}, 0)$.

4.8.2 Bayesian Game and Optimization Problem

Lemma 4.8.1 *A solution $\widehat{\mathbf{p}}$ of the optimization problem 4.2.5 is also a symmetric Bayesian equilibrium of $\mathcal{G}_I$.*

Proof. By definition (section 4.2.4, page 105), the expected aggregate surplus $W(\mathbf{p})$ is

$$W(\mathbf{p}) = \mathtt{E}_{\mathbf{g},\mathbf{m}} \left(\sum_m \widetilde{W}_m(\mathbf{g}, \mathbf{m}) \mid \mathbf{p} \right) \tag{4.8.15}$$

Let's consider the expected aggregate surplus function as a function of a profile of asymmetric application probabilities $(\mathbf{p}^1, \ldots, \mathbf{p}^I)$, where $\mathbf{p}^i = (p^i_{gm})$ is a matrix of application probabilities of worker i. The function, denoted by $W^{\text{asym}}(\mathbf{p}^1, \ldots, \mathbf{p}^I)$, is defined as

$$W^{\text{asym}}(\mathbf{p}^1, \ldots, \mathbf{p}^I) = \mathtt{E}_{\mathbf{g},\mathbf{m}} \left(\sum_m \widetilde{W}_m(\mathbf{g}, \mathbf{m}) \mid \mathbf{p}^1, \ldots, \mathbf{p}^I \right) \tag{4.8.16}$$

Maximization of the function $W(\mathbf{p})$ with respect to $\mathbf{p}$ is equivalent to maximization of the function $W^{\text{asym}}(\mathbf{p}^1, \ldots, \mathbf{p}^I)$ with respect to $\mathbf{p}^1, \ldots, \mathbf{p}^I$ under the constraints $\mathbf{p}^l = \mathbf{p}^j$ for any l and j. Let $\mathbf{p}^{-i} = (\mathbf{p}^1, \ldots, \mathbf{p}^{i-1}, \mathbf{p}^{i+1}, \ldots, \mathbf{p}^I)$ and let $\overrightarrow{\mathbf{p}} = (\mathbf{p}^1, \ldots, \mathbf{p}^I)$. The function $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ can be rewritten as

$$W^{\text{asym}}(\overrightarrow{\mathbf{p}}) = \sum_m \mathtt{E}_{g_i, m_i} \left[\mathtt{E}_{\mathbf{g}_{-i}, \mathbf{m}_{-i}} \left(\widetilde{W}_m(\mathbf{g}, \mathbf{m}) \mid g_i, m_i, \mathbf{p}^{-i} \right) \Big| \mathbf{p}^i \right] \tag{4.8.17}$$

where $\mathbf{g}_{-i} = (g_1, \ldots, g_{i-1}, g_{i+1}, g_I)$, $\mathbf{m}_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, m_I)$, $\mathtt{E}_{g_i,m_i}$ denotes expectation with respect to realizations of g_i and m_i, and $\mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}$ denotes expectation with respect to realizations of $\mathbf{g}_{-i}$ and $\mathbf{m}_{-i}$ for given distribution f and asymmetric application probabilities $\overrightarrow{\mathbf{p}}$.

Next I represent the expected surplus $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ as a sum of the expected surplus, generated by all the workers except worker i and the expected contribution of worker i to the aggregate expected surplus. Formally,

$$W^{\text{asym}}(\overrightarrow{\mathbf{p}}) = \sum_m \mathtt{E}_{g_i,m_i}\Big[\mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}\big(\widetilde{W}_m(\mathbf{g},\mathbf{m}) - \widetilde{W}_m^{-i}(\mathbf{g}_{-i},\mathbf{m}_{-i}) + \widetilde{W}_m^{-i}(\mathbf{g}_{-i},\mathbf{m}_{-i}) \mid g_i, m_i, \mathbf{p}^{-i}\big)\Big|\mathbf{p}^i\Big] = \mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}\Big(\sum_m \widetilde{W}_m^{-i}(\mathbf{g}_{-i},\mathbf{m}_{-i}) \mid \mathbf{p}^{-i}\Big) + \mathtt{E}_{g_i,m_i}\Big[\mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}\big(\widetilde{v}^i(\mathbf{g},\mathbf{m}) | g_i, m_i, \mathbf{p}^{-i}\big)\Big|\mathbf{p}^i\Big] \quad (4.8.18)$$

where $\widetilde{v}^i(\mathbf{g},\mathbf{m}) = \widetilde{W}_m(\mathbf{g},\mathbf{m}) - \widetilde{W}_m^{-i}(\mathbf{g}_{-i},\mathbf{m}_{-i})$. Suppose that $\widehat{\mathbf{p}}$ is a solution of the optimization problem

$$\max_{\mathbf{p}} W(\mathbf{p}) \quad (4.8.19)$$

and suppose that the profile of application probability matrices $(\mathbf{p}^1, \ldots, \mathbf{p}^I)$ such that $\mathbf{p}^i = \widehat{\mathbf{p}}$ for all i is not an equilibrium of the model. Then for some i there exists a deviation from strategy $\mathbf{p}^i$ to some other strategy $\widetilde{\mathbf{p}}$ that increases the expected payoff of worker i. By definition, the expected payoff of worker i is $\mathtt{E}_{g_i,m_i}\Big[\mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}\big(\widetilde{v}^i(\mathbf{g},\mathbf{m}) | g_i, m_i, \mathbf{p}^{-i}\big)\Big|\mathbf{p}^i\Big]$. Therefore, it follows from 4.8.18 that a deviation from matrix $\mathbf{p}^i$ to matrix $\widetilde{\mathbf{p}}$ increases the aggregate expected surplus $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$. Moreover, since the aggregate expected surplus $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ is a linear function with respect to $\mathbf{p}^i$

$$W^{\text{asym}}(\overrightarrow{\mathbf{p}}) = \mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}\Big(\sum_m \widetilde{W}_m^{-i}(\mathbf{g}_{-i},\mathbf{m}_{-i}) \mid \mathbf{p}^{-i}\Big) + \sum_{g_i,m_i} f_{g_i} p^i_{g_i m_i}\Big[\mathtt{E}_{\mathbf{g}_{-i},\mathbf{m}_{-i}}\big(\widetilde{v}^i(\mathbf{g},\mathbf{m}) | g_i, m_i, \mathbf{p}^{-i}\big)\Big] \quad (4.8.20)$$

the directional derivative of $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ in the direction $(0, \ldots, 0, \overbrace{\widetilde{\mathbf{p}} - \mathbf{p}^i}^{\text{element } i}, 0, \ldots, 0)$ is positive. Function $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ is symmetric with respect to $\mathbf{p}^i$. Therefore, the directional derivative of $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ in the direction $(0, \ldots, 0, \overbrace{\widetilde{\mathbf{p}} - \mathbf{p}^j}^{\text{element } j}, 0, \ldots, 0)$ is positive for any j. Since the objective function $W^{\text{asym}}(\overrightarrow{\mathbf{p}})$ is continuously differentiable with respect to $\overrightarrow{\mathbf{p}}$ the directional derivative of the function in the direction $(\widetilde{\mathbf{p}}-\mathbf{p}, \ldots, \widetilde{\mathbf{p}}-\mathbf{p})$ is also positive. Thus, the profile of application probability matrices $(\widehat{\mathbf{p}}, \ldots, \widehat{\mathbf{p}})$ is not a solution of the problem

$$\max_{(\mathbf{p}^1,\ldots,\mathbf{p}^I)} W^{\text{asym}}(\mathbf{p}^1, \ldots, \mathbf{p}^I) \qquad \text{s.t. } \mathbf{p}^l = \mathbf{p}^j \text{ for any } l, j \tag{4.8.21}$$

and, therefore, $\widehat{\mathbf{p}}$ does not maximize the function $W(\mathbf{p})$. ■

Theorem 4.3.1 *The set of solutions of the first-order conditions of the optimization problem 4.2.5 coincides with the set of symmetric Bayesian equilibria of $\mathcal{G}_I$.*

Proof. Let's consider the surplus maximization problem

$$\begin{cases} \max_{\mathbf{p}} W(\mathbf{p}) \\ \sum_m p_{gm} = 1 \\ p_{gm} \geq 0 \end{cases} \tag{4.8.22}$$

I call matrix of application probabilities $\mathbf{p}$ a feasible matrix if it satisfies constraints in 4.8.22. By definition of W^{asym}

$$W(\mathbf{p}) = W^{\text{asym}}(\mathbf{p}, \ldots, \mathbf{p}) \tag{4.8.23}$$

Suppose that $\widehat{\mathbf{p}}$ is a symmetric equilibrium of the model. Then from 4.8.20 it follows that for any feasble matrix of application probabilities $\widetilde{\mathbf{p}}$

$$W^{\text{asym}}(\mathbf{p}, \ldots, \mathbf{p}, \ldots, \mathbf{p}) \geq W^{\text{asym}}(\mathbf{p}, \ldots, \widetilde{\mathbf{p}}, \ldots, \mathbf{p}) \tag{4.8.24}$$

Therefore, the directional derivative of W^{asym} in the direction $(0, \ldots, 0, \widetilde{\mathbf{p}} - \mathbf{p}, 0, \ldots, 0)$ is non-negative. Thus, directional derivative of W in the direction

$\widetilde{\mathbf{p}} - \mathbf{p}$ is non-negative for any feasible matrix $\widetilde{\mathbf{p}}$. This is true only if the first-order conditions hold at the point $\mathbf{p}$.

Suppose now that the first-order conditions are satisfied at point $\mathbf{p}$. Then the directional derivative of the function W is nondecreasing in any direction $\widetilde{\mathbf{p}} - \mathbf{p}$. Note that either any i the directional derivatives of W^{asym} in the direction $(0, \dots, 0, \widetilde{\mathbf{p}} - \mathbf{p}, 0, \dots, 0)$, where $\widetilde{\mathbf{p}} - \mathbf{p}$ is at i^{th} position, are nonnegative or all are positive. From 4.8.23 it follows that only the first possibility can be true. Since the function W^{asym} is linear in each $\mathbf{p}^i$ the function W^{asym} is nondecreasing in the direction $(0, \dots, 0, \widetilde{\mathbf{p}} - \mathbf{p}, 0, \dots, 0)$. Therefore, $\mathbf{p}$ is an equilibrium. ■

Corollary 4.3.2 *If the expected aggregate surplus function $W(\mathbf{p})$ is concave then any symmetric equilibrium of $\mathcal{G}_I$ maximizes $W(\mathbf{p})$. If the function $W(\mathbf{p})$ is* <u>*strictly*</u> *concave then there exists a unique symmetric equilibrium of $\mathcal{G}_I$.*

Proof. The corollary follows immediately from the theorem 4.3.1. ■

4.8.3 Properties of Optimization Problem

Discrete Model

Note that there is a linear relationship $h_g = f_g p_{gm}$ between distribution h and application probabilities $\mathbf{p}$. Therefore, concavity of the function $\mathtt{E} \sum_{i=1}^{K_m} X^{(i)}$ with respect to h implies concavity of the function $W_m(\mathbf{p}_m)$ with respect to $\mathbf{p}_m$.

Proposition 4.8.3 *Let $X_1, X_2, \dots, X_I$ be independent identically distributed random variables. Let $\{\omega_0, \omega_1, \dots, \omega_G\}$ be a support of the random variables and let $h = (h_0, h_1, \dots, h_G)$ (where $h_g = Pr(X_i = \omega_g)$) denote a common* *distribution of the random variables. Then for any k, the function $V^k(h) = \mathtt{E}\sum_{i=1}^{k} X^{(i)}$ is a concave function of distribution h.*

Proof. Let $\mu_{i,n} = \mathtt{E}X^{(i)}$, where $X^{(i)}$ is $i^{\mathtt{th}}$ order statistics constructed from n i.i.d random variables $X_1, X_2, \ldots, X_n$. Let $h^a = (h^a_0, \ldots, h^a_G)$ and $h^b = (h^b_0, \ldots, h^b_G)$ be two arbitrary distributions with support $\{\omega_0, \omega_1, \ldots, \omega_G\}$ and $h(\lambda) = \lambda h^a + (1-\lambda)h^b$, $\lambda \in [0,1]$. Let also H^a, H^b and $H(\lambda)$ be the cumulative distribution functions that corresponds to distributions h^a, h^b and $h(\lambda)$. Given $h(\lambda)$ I find the expectations $\mu_{i,n}$, $i = k \ldots n$. I need to show that the function $\sum_{i=k\ldots n} \mu_{i,n}$ is a concave function of $\lambda \in [0,1]$ for an arbitrary choice of distributions h^a and h^b. The proof is shown in two steps:

(a) $\mu_{i,n} = \sum_{j=i\ldots n} \binom{j-1}{i-1}\binom{n}{j}(-1)^{j-i}\mu_{j,j}$

reference: H.A. David [10],

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ and $\mu_{j,j} = \sum_{g=0}^{G}(\omega_{g+1} - \omega_g)(1 - [H_g(\lambda)]^j)$. (The function $[H_g(\lambda)]^j$ is a cumulative distribution function of the $j^{\mathtt{th}}$ order statistic $X^{(j)}$ if the sequence size is j). I need to show that the following function is negative:

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}\sum_{i=k}^{n}\mu_{i,n} = &-\sum_{g=0}^{G}(\omega_{g+1} - \omega_g)\left[H^a_g - H^b_g\right]^2 \times \\ &\times \sum_{i=k}^{n}\sum_{j=i}^{n} j(j-1)\binom{n}{j}\binom{j-1}{i-1}(-1)^{j-i}\left(H_g(\lambda)\right)^{j-2} \leq 0 \quad (4.8.25)\end{aligned}$$

(b) $\phi_{k,n}(s) =$

$= \sum_{i=k}^{n}\sum_{j=i}^{n} j(j-1)\binom{n}{j}\binom{j-1}{i-1}(-1)^{j-i}s^{j-2}$ is nonnegative for any k, n, and $s \in [0,1]$. This is shown in the following lemma.

■

Lemma 4.8.2

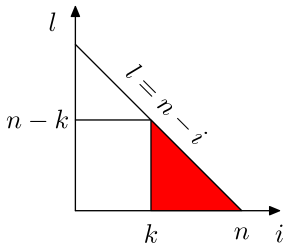
$$\phi_{k,n}(s) = \frac{n!}{(k-2)!(n-k)!} s^{k-2}(1-s)^{n-k} \tag{4.8.26}$$

Proof.

$$\phi_{k,n}(s) = \sum_{i=k}^{n}\sum_{j=i}^{n} j(j-1)\frac{n!}{j!(n-j)!}\frac{(j-1)!}{(i-1)!(j-i)!}(-1)^{j-i}s^{j-2} =$$

$$= \sum_{i=k}^{n}\sum_{j=i}^{n} \frac{n!}{i!(j-i)!(n-j)!}(j-1)i(-1)^{j-i}s^{j-2} =$$

$$\text{denote } l = j-i \text{ and change summation order} = \sum_{l=0}^{n-k}\sum_{i=k}^{n-l} \frac{n!}{i!(l)!(n-l-i)!} i(l+i-1)(-1)^l s^{l+i-2} \tag{4.8.27}$$



The picture shows the (i,l) region (filled with red) over which the summation takes place

Let

$\varphi_{k,n}(s) = \sum_{l=0}^{n-k}\sum_{i=k}^{n-l} \frac{n!}{i!(l)!(n-l-i)!} i(-1)^l s^{l+i-1}$,
$\varphi_{k,n}(s,t) = \sum_{l=0}^{n-k}\sum_{i=k}^{n-l} \frac{n!}{i!(l)!(n-l-i)!} i(-1)^l s^{i-1}t^l$,
$\zeta_{k,n}(s,t) = \sum_{l=0}^{n-k}\sum_{i=k}^{n-l} \frac{n!}{i!(l)!(n-l-i)!} (-1)^l s^i t^l$

The following equalities hold

$$\phi_{k,n}(s) = \tfrac{\partial}{\partial s}\varphi_{k,n}(s),$$
$$\varphi_{k,n}(s) = \varphi_{k,n}(s,s),$$
$$\varphi_{k,n}(s,t) = n\zeta_{k-1,n-1}(s,t)$$

The first two equations are trivial. The last equation holds because $\varphi_{k,n}(s,t) = n\sum_{l=0}^{(n-1)-(k-1)}\sum_{i-1=k-1}^{n-1-l}\frac{(n-1)!}{(i-1)!(l)!((n-1)-l-(i-1))!}(-1)^l s^{i-1}t^l$.

Taking derivative of $\zeta_{k,n}(s,t)$ with respect to t and s I get the following identities:

$$\frac{\partial\zeta_{k,n}(s,t)}{\partial t} = \sum_{l=1}^{n-k}\sum_{i=k}^{(n-1)-(l-1)}\frac{n(n-1)!}{i!(l-1)!(n-1-i-(l-1))!}(-1)(-1)^{l-1}s^i t^{l-1} = -n\zeta_{k,n-1}$$
$$\frac{\partial\zeta_{k,n}(s,t)}{\partial s} = \sum_{l=0}^{n-k}\sum_{i=k}^{n-l}\frac{n(n-1)!}{(i-1)!l!(n-1-(i-1)-l)!}(-1)^l s^{i-1}t^l = n\zeta_{k-1,n-1}$$

Therefore

$$\begin{aligned}\phi_{k,n}(s) = \frac{\partial}{\partial s}\varphi_{k,n}(s,s) &= \frac{\partial}{\partial s}\varphi_{k,n}(s,t)|_{t=s} + \frac{\partial}{\partial t}\varphi_{k,n}(s,t)|_{t=s} = \\ &= n\frac{\partial}{\partial s}\zeta_{k-1,n-1}(s,t)|_{t=s} + \frac{\partial}{\partial t}\zeta_{k-1,n-1}(s,t)|_{t=s} = \\ &= n(n-1)(\zeta_{k-2,n-2}(s,s) - \zeta_{k-1,n-2}(s,s))\end{aligned}$$

Finally,

$$\zeta_{k+1,n}(s,s) = \sum_{l=0}^{n-k-1} \sum_{i=k+1}^{n-l} \frac{n!}{i!(l)!(n-l-i)!}(-1)^l s^{l+i} =$$

$$\left[\begin{array}{l}\text{increase the upper index in the } 1^{\text{st}} \text{ sum} \\ \text{decrease the lower index in the } 2^{\text{nd}} \text{ sum}\end{array}\right]$$

$$= \sum_{l=0}^{n-k} \left[\sum_{i=k}^{n-l} \frac{n!}{i!(l)!(n-l-i)!}(-1)^l s^{l+i} - \frac{n!}{k!l!(n-k-l)!}(-1)^l s^{k+l} \right] - \sum_{i=k+1}^{k} (\ldots) =$$

$$= \zeta_{k,n}(s,s) - \sum_{l=0}^{n-k} \frac{n!}{k!l!(n-k-l)!}(-1)^l s^{k+l} =$$

$$= \zeta_{k,n}(s,s) - \frac{n!}{k!(n-k)!} s^k \sum_{l=0}^{n-k} \frac{(n-k)!}{l!(n-k-l)!}(-1)^l s^l =$$

$$= \zeta_{k,n}(s,s) - \frac{n!}{k!(n-k)!} s^k (1-s)^{n-k} \quad (4.8.28)$$

which proves the lemma. ■

Proposition 4.8.4 *Let's define function $V(p)$ as follows. There is an infinite staircase and a lady standing at the bottom of the staircase. The lady flips a coin I times. Each time the coin shows heads, the lady goes one step up. Each time the coin shows tails, she does not move. The probability of heads is p. Function $V(p)$ is defined as expected height that the lady goes up. Then $V(p)$ is a concave function of p if the height of the stairs is decreasing with the height and is a convex function of p if the height of the stairs is increasing with the height.*

Proof. Let V_i be cumulative height of i steps, $(V_0 = 0, V_n = V)$ and $v_i = V_i - V_{i-1}$ be the height of step i. By assumption v_i is decreasing with i. The expected height that the lady goes up is equal to $W(p) = \sum_{i=0}^{n} C_n^i p^i (1-p)^{n-i} V_i = V - \sum_{i=0}^{n} v_i P_i^{(1)} = V + \sum_{i=0}^{n} \triangle v_i P_i^{(2)}$, where $P_i^{(1)} = \sum_{s=0}^{i} C_n^s p^s (1-p)^{n-s}$, $P_i^{(2)} = \sum_{s=0}^{i} P_s^{(1)}$, and $\triangle v_i = v_{i+1} - v_i \leq 0$.

To finish the proof I have to show that $P^{(2)}(p)$ is a convex function. This is done in the following lemma. ■

Lemma 4.8.3 *$\frac{\partial P^{(1)}_{i,n}}{\partial p} = -np_{i,n-1}$ or, more generally, $\frac{\partial P^{(k)}_{i,n}}{\partial p^k} = (-1)^k \frac{n!}{(n-k)!} p_{i,n-k}$.*

where $p_{i,n}$ is the distribution of the Binomial $B(p,n)$ distribution and $P^{(k)}_{i,n}$ is the $k^{\mathtt{th}}$ cumulative sum of the binomial $B(p,n)$ distribution.

Proof. This is proved by taking directly the derivative of $P^{(1)}_{i,n} = \sum_{s=0}^{i} C_n^s p^s (1-p)^{n-s}$ with respect to p. ■

Theorem 4.2.1 *Suppose that either*

(a) *Firm $m \in \{1, ...M\}$ has a constant reserve value of each job position (that is, for each m, $r_{jm} = r_m$ for all j)*

or

(b) *The productivity of a worker does not depend on the type of the worker at firm $m \in \{1, ...M\}$ (that is, for each m, $\omega_{gm} = \omega_m$ for any g)*

Then the expected surplus function $W(\mathbf{p})$ is a concave function of a $G \times M$ matrix of application probabilities $\mathbf{p}$ on the set of all feasible matrices of application probabilities.

Proof. If restriction (a) holds then the function $W_m(\mathbf{p})$ can be represented as described in Proposition 4.8.1. The concavity of the function follows then from Proposition 4.8.3.

If restriction (b) holds then the function $W_m(\mathbf{p})$ can be represented as described in Proposition 4.8.2. The concavity of the function follows then from Proposition 4.8.4.

Concavity of $W_m(\mathbf{p})$ implies immediately concavity of $W(\mathbf{p}) = \sum_m W_m(\mathbf{p})$.

■

Continuous Model

Lemma 4.6.1 *If Assumption 4.6.1 holds then the function $W^c_m(\mathbf{q}_m)$ is strictly concave and, therefore, for any vector $u = (u_1, \ldots, u_G)$ there exists a unique solution $\widehat{\mathbf{q}}_m$ to the optimization problem*

$$\max_{\mathbf{q}_m \geq 0} \left[W^c_m(\mathbf{q}_m) - \sum_g u_q q_{gm} \right] + \sum_g \alpha_g u_g \tag{4.8.29}$$

Proof. Without loss of generality, let's assume that the productivity of workers at a firm of type m are ordered as follows

$$\omega_{1m} > \omega_{2m} > \ldots > \omega_{Gm} > r_m \tag{4.8.30}$$

so that the function $W^c_m(q_{1m}, \ldots, q_{Gm})$ can be written as follows

$$\begin{aligned} W^c_m(q_{1m}, \ldots, q_{Gm}) = \\ = \omega_{1m} - (\omega_{1m} - \omega_{2m}) \exp(-q_{1m})) - (\omega_{2m} - \omega_{3m}) \exp(-q_{1m} - q_{2m})) - \\ - \ldots - \omega_{Gm} \exp(-q_{1m} - \ldots - q_{Gm})) \end{aligned} \tag{4.8.31}$$

Let's make a change of variable $\mathbf{x}_m = A\mathbf{q}_m$ in the function $W^c_m(\mathbf{q}_m)$, where matrix A has ones on the main diagonal and everywhere below the main diagonal and matrix A has zeros everywhere above the main diagonal. The function $W^c_m(\mathbf{q})$ can be written as $W^c_m(\mathbf{q}_m) = v_m(A\mathbf{q}_m)$, where

$$\begin{aligned} v_m(\mathbf{x}_m) = \omega_{1m} - (\omega_{1m} - \omega_{2m}) \exp(-x_{1m}) - (\omega_{2m} - \omega_{3m}) \exp(-x_{2m}) - \\ \ldots - \omega_{Gm} \exp(x_{Gm}) \end{aligned} \tag{4.8.32}$$

The second derivative of $W^c_m(\mathbf{q}_m)$ with respect to $\mathbf{q}_m$ is equal to

$$\frac{\partial^2 W^c_m}{\partial \mathbf{q}^2_m} = A^T \frac{\partial^2 v_m}{\partial \mathbf{x}^2_m}(A\mathbf{q}_m)A \tag{4.8.33}$$

Strict concavity of $W^c_m(\mathbf{q}_m)$ follows from equation 4.8.33, strict concavity of the function $v_m(\mathbf{x}_m)$, and non-singularity of matrix A.

Uniqueness of the solution of optimization problem 4.8.29 follows from strict concavity of the function $W^c_m(\mathbf{q}_m)$. ■

Lemma 4.8.4 *Suppose that Assumption 4.6.1 holds. Let* $[D^2W^c_m]^{-1}$ *be the inverse of the second derivative matrix of* $W^c_m(\mathbf{q}_m)$*. Then the matrix* $[D^2W^c_m]^{-1}$ *has the following properties. Matrix* $[D^2W^c_m]^{-1}$ *has negative elements on the main diagonal and positive elements on the diagonals below and above the main diagonals. All the other elements are zero. The sum of the elements along any row or column is smaller or equal to zero.*

Proof. Let's make the same change of variables as in the proof of Lemma 4.6.1. It can be directly verified that the matrix A^{-1} has ones on the main diagonal, minus ones on the diagonal below the main and zeros everywhere else. The second derivative of $W^c_m(\mathbf{q}_m)$ with respect to $\mathbf{q}_m$ is given in 4.8.33. Therefore, the inverse of $\frac{\partial^2 W^c_m}{\partial \mathbf{q}^2}$ is equal to

$$\frac{\partial^2 W^c_m}{\partial \mathbf{q}^2_m}^{-1} = A^{-1}\Lambda_m(A^T)^{-1} \tag{4.8.34}$$

where the matrix

$$\Lambda_m = \left(\frac{\partial^2 v_m}{\partial \mathbf{x}^2_m}\right)^{-1}(A\mathbf{q}_m) \tag{4.8.35}$$

has negative elements $-\lambda_{gm} = [-\exp(-x_{gm})(\omega_{gm} - \omega_{g+1m})]^{-1}$ on the main diagonal and zero off-diagonal elements. Multiplying the matrices in 4.8.34, I obtain the matrix that has the following elements in the g^{th} row, $g = 1, \ldots, G$. The element at position g is $-\lambda_{g-1m} - \lambda_{gm}$, the elements at positions $g-1$ and $g+1$ are λ_{g-1m} and λ_{gm}, and the element at any other position is zero. The sum of the elements in each row and column is equal to zero except the last row and column where the sum is smaller or equal to zero. ■

Lemma 4.6.2 *The solution* $q_{gm}(u)$ *of the optimization problem 4.8.29 has the following properties.*

1. *For each g, the function $q_{gm}(u_1, \ldots, u_G)$ is a non-increasing function of u_g.*

2. *For each g, the function $q_{gm}(u_1, \ldots, u_G)$ is a non-decreasing function of u_h for any $h \neq g$.*

Proof. Let $\mathbf{u} = (u_1, \ldots, u_G)$ and $\mathbf{q}_m(\mathbf{u}) = (q_{1m}(\mathbf{u}), \ldots, q_{Gm}(\mathbf{u}))$. The function $\mathbf{q}_m(\mathbf{u})$ is the solution of the following optimization problem.

$$\max_{\mathbf{q}_m \geq 0} \left[W_m^c(\mathbf{q}_m) - \sum_l u_l q_{lm} \right] \tag{4.8.36}$$

Consider first the case that $q_{gm}(\mathbf{u}) = 0$. As u_h increases, the function $q_{gm}(\mathbf{u})$ does not decreasing since $q_{gm}(\mathbf{u}) \geq 0$. Suppose now that u_g increases. The objective function in 4.8.36 decreases for all positive values of q_{gm} and does not change if $q_{gm} = 0$. Therefore, the solution of 4.8.36 does not change with an increase in u_g.

Suppose now that $q_{gm}(\mathbf{u}) > 0$. Let γ_{gm} denote lagrange multipliers associated with constraints $q_{gm} \geq 0$ and let $\Gamma_m^+ = \{l : \gamma_{lm}(\mathbf{u}) = 0\}$. Without loss of generality, let's assume that $\Gamma_m^+ = \{1, 2, \ldots, G_0\}$. Let $\mathbf{q}_m^+ = (q_{lm}, \ldots, q_{G_0 m})$, $\mathbf{u}^+ = (u_{lm}, \ldots, u_{G_0 m})$, and $W_m^{c,+}(\mathbf{q}_m^+) = W_m^c(q_{1m}, \ldots, q_{G_0 m}, 0 \ldots, 0)$. Let's consider the following unconstrained optimization problem

$$\max_{\mathbf{q}_m} \left[W_m^{c,+}(\mathbf{q}_m^+) - \sum_{l \in \Gamma_m^+} u_l^+ q_{lm}^+ \right] \tag{4.8.37}$$

Note that the functional form of $W_m^c(\mathbf{q}_m)$, given in 4.8.31, and $W_m^{c,+}(\mathbf{q}_m^+)$ is similar and Assumption 4.6.1 holds in the case of $W_m^{c,+}(\mathbf{q}_m^+)$. Therefore, Lemma 4.8.4 applies to $W_m^{c,+}(\mathbf{q}_m^+)$. The solution of 4.8.37, denoted by $\mathbf{q}_m^+(\mathbf{u}^+)$, can be found from the first-order conditions

$$\frac{dW_m^{c,+}}{d\mathbf{q}_m^+} = \mathbf{u}^+ \tag{4.8.38}$$

Taking derivative with respect to $\mathbf{u}^+$ of the left- and right-hand side in 4.8.38 I obtain the following expression for $\frac{d\mathbf{q}_m^+}{d\mathbf{u}^+}$

$$\frac{d\mathbf{q}_m^+}{d\mathbf{u}^+} = \left[D^2 W_m^{c,+}\right]^{-1} \tag{4.8.39}$$

By Lemma 4.8.4, the derivative of $\mathbf{q}_{gm}^+$ is negative with respect to g and positive with respect to any $h \neq g$. Therefore, if u_g increases by some small ϵ then $\mathbf{q}_{gm}^+$ decreases and $\mathbf{q}_{hm}^+$ increases for any $h \neq g$. The new solution is feasible and, therefore, the vector $\mathbf{q} = (\mathbf{q}^+, 0)$ is the solution of 4.8.36. ■

4.8.4 Convergence

The convergence theorems are quite technical.

Proposition 4.5.1 *Suppose that a set of firms $\mathcal{F}$ is identical from the point of view of all workers. That is, the number of jobs, the reserve values of the jobs, and the productivity of a worker of type $g \in \{1, \ldots, G\}$ is the same at each firm in the set. Then there exists a surplus maximizing matrix of application probabilities such that for each g workers of types g apply with equal probabilities to the firms in the set.*

Proof. Let's assume without loss of generality that the set of the identical firms is $\mathcal{F} = \{1, \ldots, M_1\}$. Let p_{gm} denote the surplus maximizing application probabilties. Let also Σ denote the set of all possible permutations of the elements $(1, 2, \ldots, M_1)$. Because the firms $1, \ldots, M_1$ are identical the application probabilties $p_{gm}^\sigma = p_{g\sigma(m)}$ for $m = 1, \ldots, M_1$ and $p_{gm}^\sigma = p_{gm}$ for $m = M_1 + 1, \ldots, M$ are optimal for an arbitrary permutation $\sigma \in \Sigma$. Since the expected surplus functional is concave I obtain $W(\frac{1}{M_1!}\sum_{\sigma\in\Sigma} p^\sigma) \geq \frac{1}{M_1!}\sum_{\sigma\in\Sigma} W(p^\sigma) = W(p)$. Therefore, the matrix of application probabilties $\frac{1}{M_1!}\sum_{\sigma\in\Sigma} p^\sigma$ maximizes the surplus and the firms $m = 1, \ldots, M_1$ are selected with the same probability. ■

Theorem 4.6.1

1. *Suppose that* $\mathbf{p}$ *and* $\mathbf{q}$ *are such that*

$$p_{gm} = \frac{\beta_m}{\alpha_g} q_{gm} \tag{4.8.40}$$

Then $W_m^{(n)}(\mathbf{p}) \to W_m^c(\mathbf{q})$ *and, therefore,* $\frac{W^{(n)}(\mathbf{p})}{n} \to W^c(\mathbf{q})$ *as* $n \to \infty$*, where the convergence is pointwise.*

2. *Suppose that the functions* $W^{(n)}(\mathbf{p})$ *are concave with respect to* $\mathbf{p}$ *for any* n*. Then the function* $W^c(\mathbf{q})$ *is concave with respect to* $\mathbf{q}$.

Proof. The general intuition is standard. Informally, it says that the sum of a large number of arbitrary distributed random variables each of which is close to zero is distributed approximately as a Poisson distribution. There obviously exist general theorems that formalize the idea. I show this directly without making reference to any results.

By the central limit theorem the number of workers of type g in a large finite model is distributed as $\widetilde{\alpha}_g \simeq \alpha_g n + \sqrt{n} Z_g$, where Z_g is some gaussian random vector. The choice of a firm of type m by a worker of type g is a random variable Y_{gm}^i with a distribution $Y_{gm}^i = \begin{cases} 1 & \text{with probability } \frac{p_{gm}}{n} \\ 0 & \text{with probability } 1 - \frac{p_{gm}}{n} \end{cases}$. $Y_{gm}^i = 1$ if the worker selects the firm and zero otherwise. The characteristic function of Y_{gm}^i is $\varphi_{gm}^{(n)}(z) = 1 - \frac{p_{gm}}{n}(1-z)$. (The characteristic function is defined as $\varphi_{gm}^{(n)}(z) = \mathtt{E} z^{Y_{gm}^i}$).

The number of the workers that apply to the firm m is a random variable $\zeta_{gm}^{(n)} = \sum_{i=1}^{\widetilde{\alpha}_g} Y_{gm}^i$. Let $\phi_{gm}^{(n)}(z)$ be the characteristic function of $\zeta_{gm}^{(n)}$. I have $\phi_{gm}^{(n)}(z) = E_{\widetilde{\alpha}_g}(\varphi_{gm}^{(n)})^{\widetilde{\alpha}_g}$. As $\frac{\widetilde{\alpha}_g}{n} \to \alpha_g$ I have $\lim_{n\to\infty} \phi_{gm}^{(n)}(z) = \lim_{n\to\infty} \left[1 - \frac{p_{gm}}{n}(1-z)\right]^{n\alpha_g} = \exp(-q_{gm}(1-z))$. But the function $\exp(-q_{gm}(1-z))$ is the characteristic function of the Poisson distribution with parameter q_{gm}.

Since all the functions that I consider are continuous and bounded the weak convergence of the distributions guarantee the convergence of the expectations of the surplus functionals. ■

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